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Curves and Patches

The circle

$$x^2 + y^2 = R^2 \qquad \text{where } R \text{ is the radius.}$$

Although this equation has its uses, it is not very convenient for drawing the curve. What we really want are two functions that determine the coordinates of any point on the circumference in terms of some parameter.

$$x = R\cos(t)$$

$$y = R\sin(t) \qquad 0 \le t \le 2\pi$$

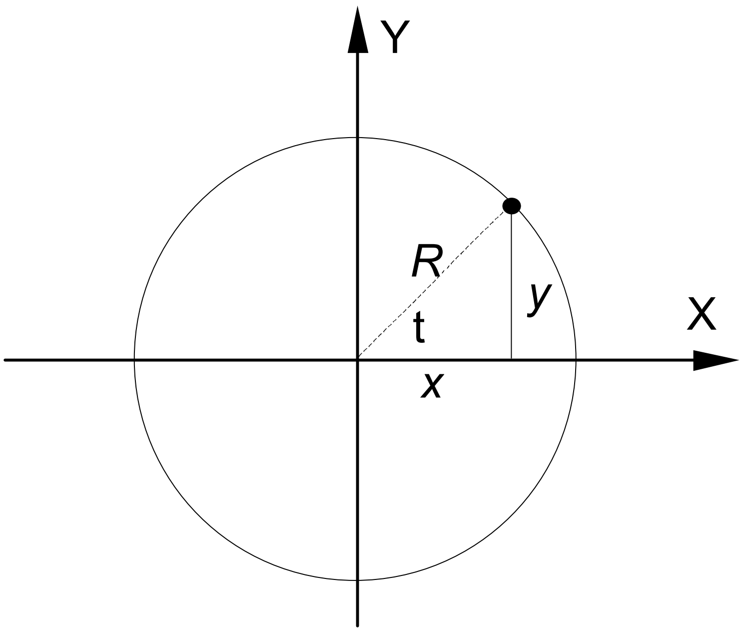

Fig. 9.1 *The circle can be drawn by tracing out a series of points on the*

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The ellipse

$$\frac{x^2}{R_{maj}^2} + \frac{y^2}{R_{\min}^2} = 1$$

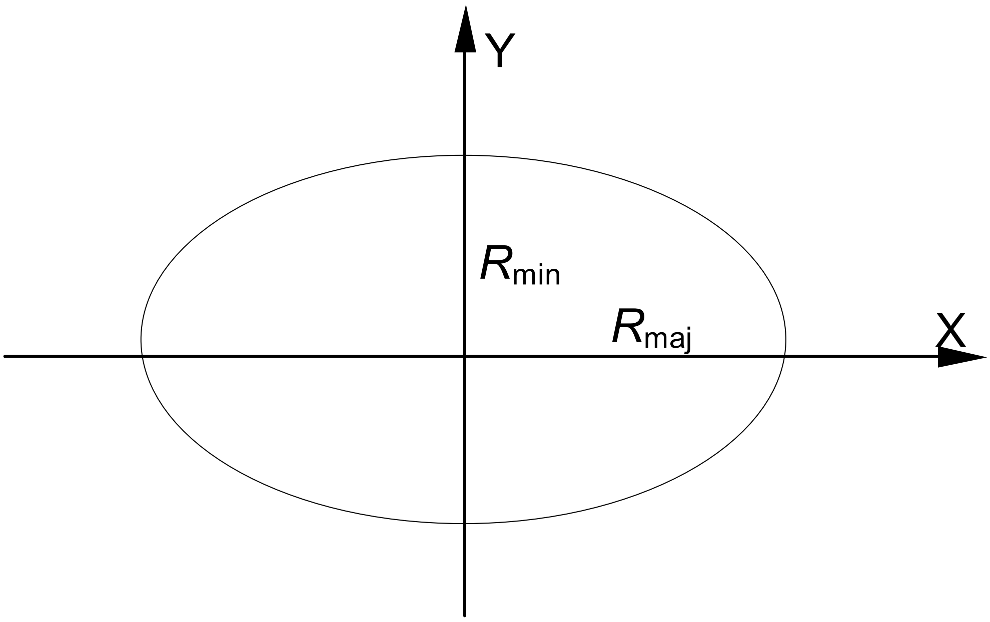

Fig. 9.2 *An ellipse showing the major and minor radii*

Bézier curves

Two people are associated with what are now called Bézier curves: Paul de Casteljau, who worked at Citroën and Pierre Bézier, who worked at Rénault.

De Casteljau's work was slightly ahead of Bézier, but because of Citroen's policy of secrecy it was never published, and Bézier's name has since been associated with the theory of polynomial curves and surfaces.

Casteljau started his research work in 1959, and his reports were only discovered in 1975, by which time Bézier had already become known for his special curves and surfaces.

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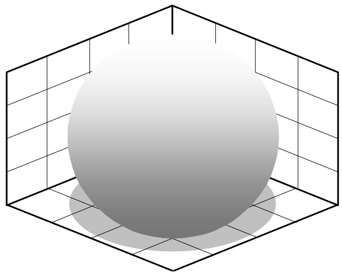

Binomial expansion

The expansion of $(x+a)^n$ for different values of n is

$$(x+a)^0 = 1$$

$$(x+a)^1 = 1x + 1a$$

$$(x+a)^2 = 1x^2 + 2ax + 1a^2$$

$$(x+a)^3 = 1x^3 + 3ax^2 + 3a^2x + 1a^3$$

$$(x+a)^4 = 1x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + 1a^4$$

Table 9.1 Pascal's triangle.

	i						
n	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

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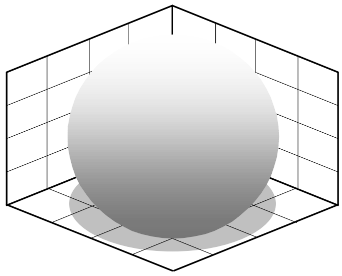


Bernstein polynomial terms

If $(x+a)^2 = x^2 + 2ax + a^2$

Then if $x = (1 - t)$ and $a = t$

$$\therefore ((1-t)+t)^2 = (1-t)^2 + 2(1-t)t + t^2$$

But as $(1 - t) + t = 1$

$$\therefore ((1-t)+t)^2 = (1-t)^2 + 2(1-t)t + t^2 = 1$$

Therefore it can be used as an interpolant.

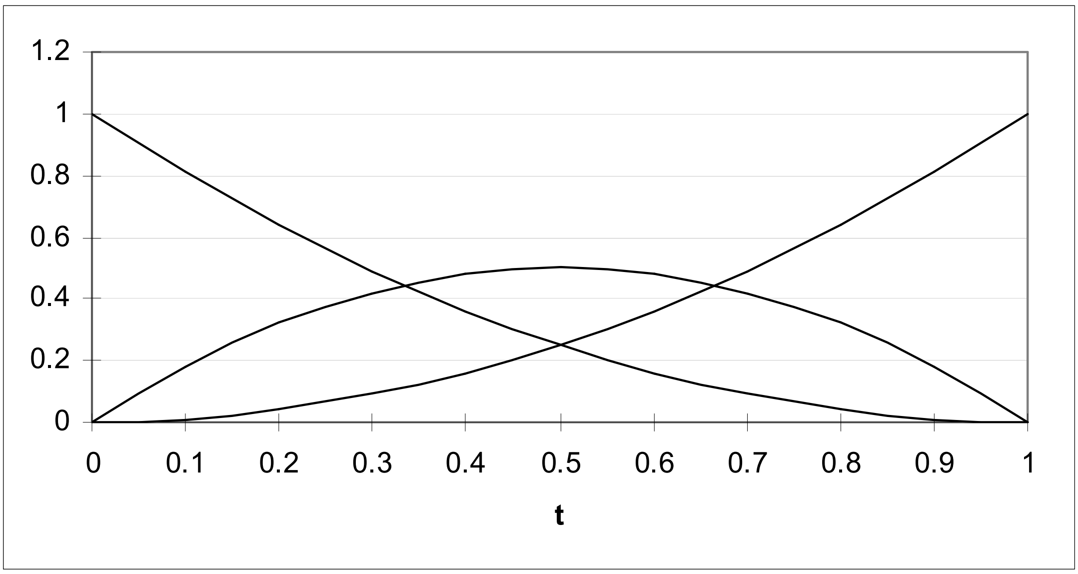


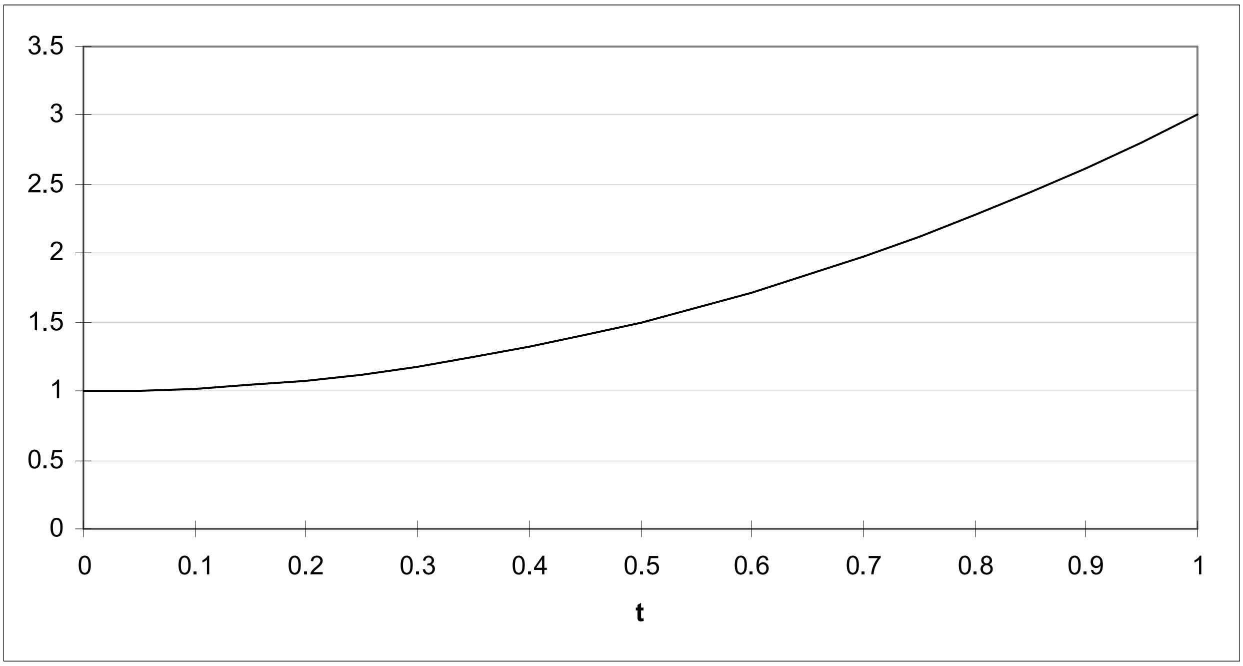
***Fig. 9.3** The graphs of the quadratic Bernstein polynomials*

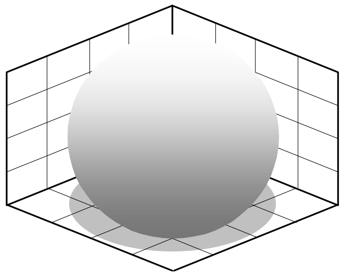
Interpolating

We can interpolate between V_1 and V_2 as follows

$$V = V_1(1-t)^2 + 2t(1-t) + V_2 t^2$$

If $V_1 = 1$ and $V_2 = 3$ we obtain the following curve





Interpolating

There is nothing preventing us from multiplying the middle term $2t(1-t)$ by any arbitrary number V_c

$$V = V_1(1-t)^2 + V_c 2t(1-t) + V_2 t^2$$

For example, if $V_c = 3$ we obtain the following graph

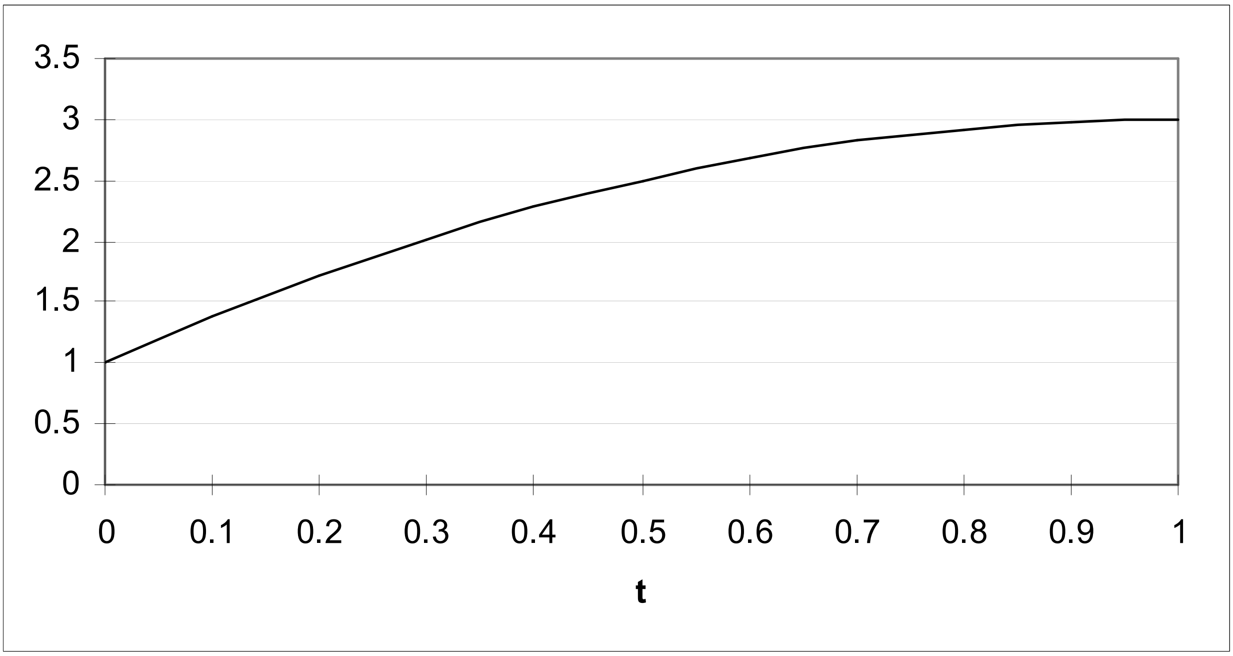


Fig. 9.5 *Bernstein interpolation between values 1 and 3 with V_c = 3.*

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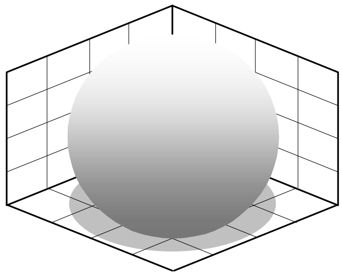


Interpolating

$$V = V_1(1-t)^2 + V_c 2t(1-t) + V_2 t^2$$

But $V_c = 3$ can be anything

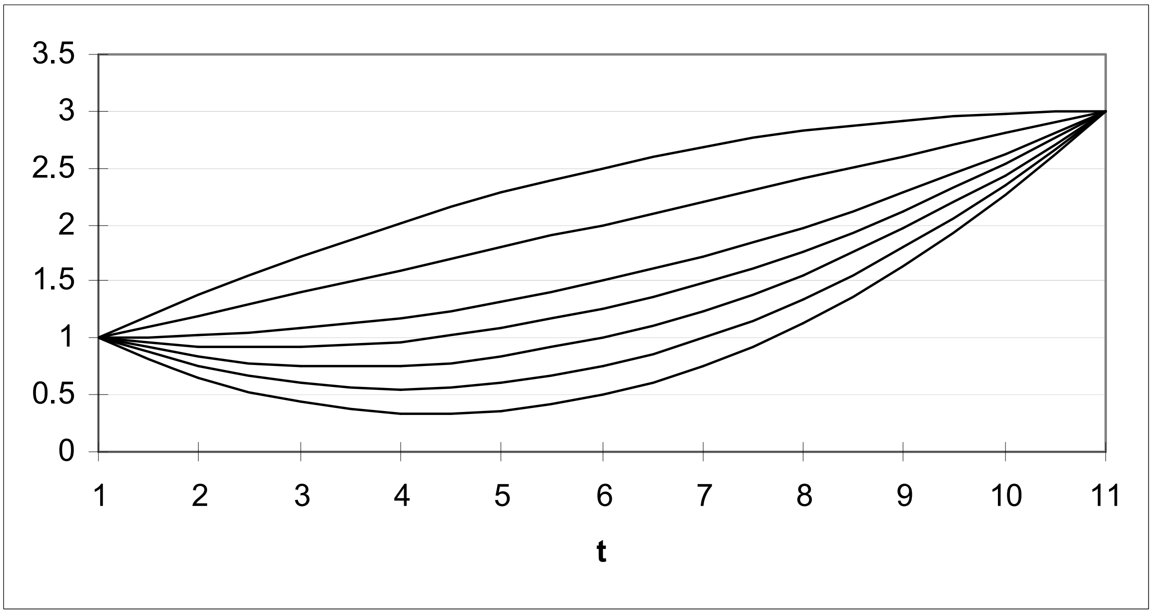
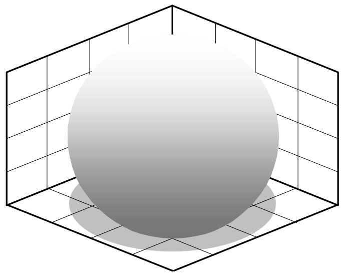


Fig. 9.6 *Bernstein interpolation between values 1 and 3 for different values of V_c.*



Linear interpolation

If V_c is $\frac{1}{2}(V_1 + V_2)$ we obtain linear interpolation

$$V = 1(1-t)^2 + 2\times 2t(1-t) + 3t^2$$
$$V = 1 - 2t + t^2 + 4t - 4t^2 + 3t^2$$
$$V = 1 + 2t$$

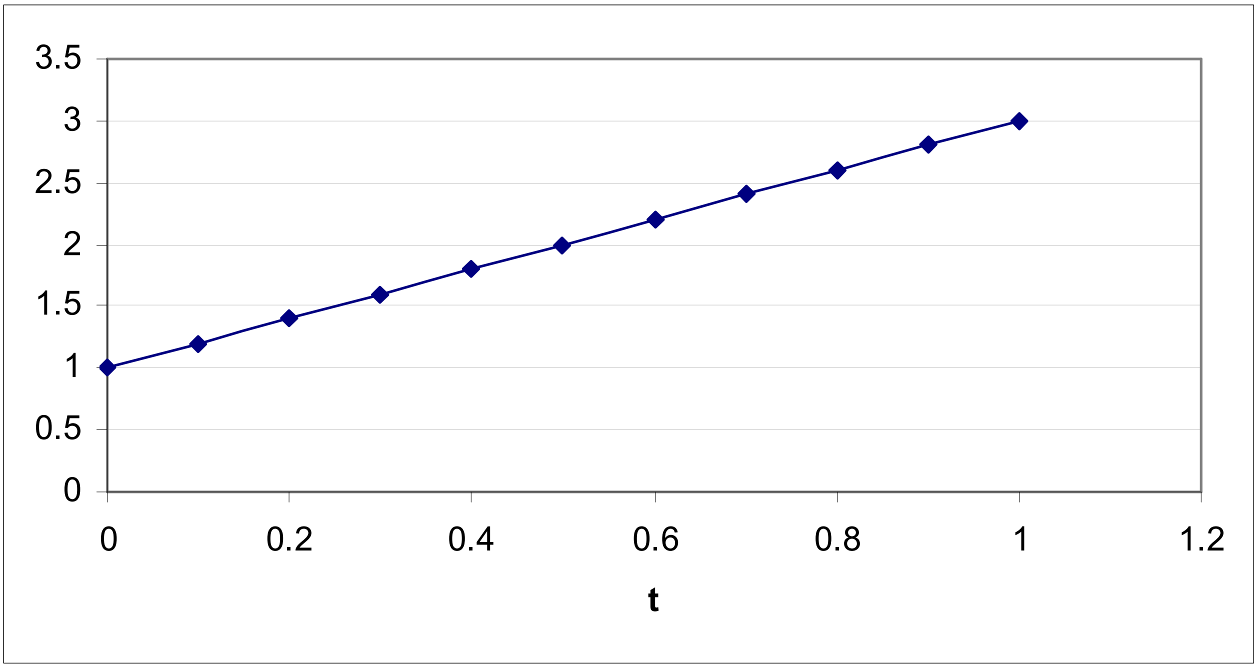


Fig. 9.7 *Linear interpolation using a quadratic Bernstein interpolant.*

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Quadratic Bézier curves

Quadratic Bézier curves are formed by using Bernstein polynomials to interpolate between the *x*-, *y*- and *z*-coordinates associated with the start and end points forming the curve.

We can draw a 2D quadratic Bézier curve between (1, 1) and (4, 3) using the following equations

$$x = 1(1-t)^2 + x_c 2t(1-t) + 4t^2$$
$$y = 1(1-t)^2 + y_c 2t(1-t) + 3t^2$$

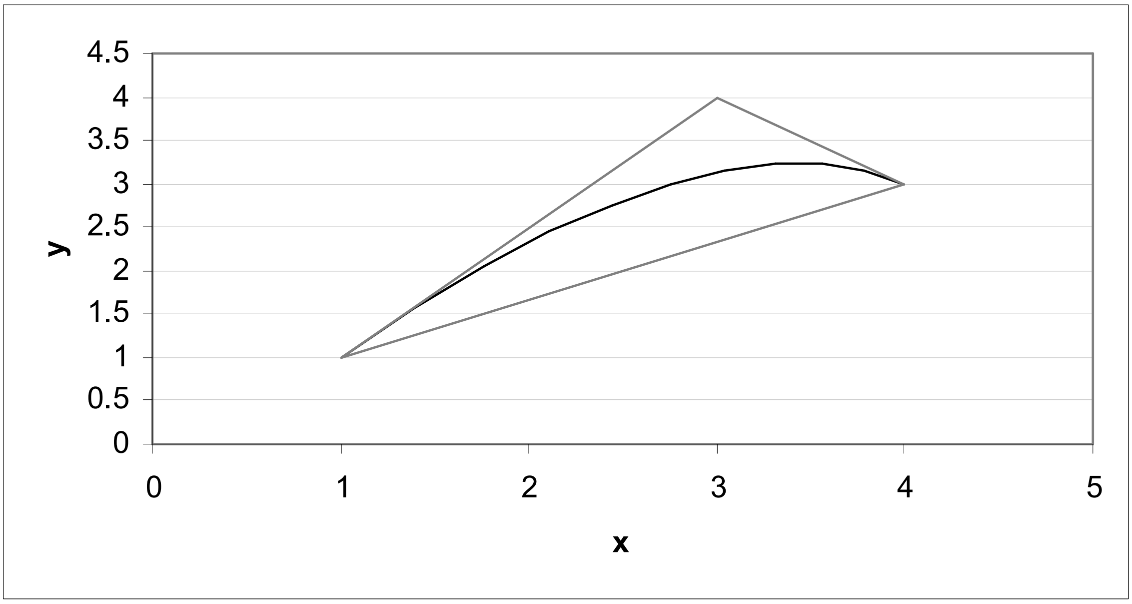


Fig. 9.8 *Quadratic Bezier curve between (1, 1) and (4, 3), with (3, 4) as the control vertex.*

Quadratic Bézier Curve

A Bézier curve possess interpolating and approximating qualities: the interpolating feature ensures that the curve passes through the end points, whilst the approximating feature shows how the curve passes close to the control point.

Convex Hull

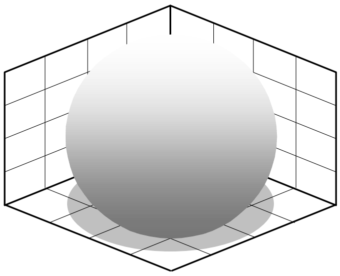
The convex hull property of Bézier curves means that the curve is always contained within the polygon connecting the end and control points.

The slope of the curve at (1, 1) is equal to the slope of the line connecting the start point to the control point (3, 4).

And the slope of the curve at (4, 3) is equal to the slope of the line connecting the control point (3, 4) to the end point (4, 3).

There are no restrictions placed upon the position of (x_c, y_c) - it can be anywhere.

3D curves are created by including z-coordinates for the start, end and control vertices.



Cubic Bernstein polynomials

A cubic curve supports one peak and one valley, which simplifies the construction of more complex curves.

The cubic equation becomes

$$\left((1-t)+t\right)^3 = (1-t)^3 + 3t(1-t)^2 t + 3t^2(1-t) + t^3$$

Which can be used as an interpolant as

$$V = V_1(1-t)^3 + V_{c1}3t(1-t)^2 t + V_{c2}3t^2(1-t) + V_2 t^3$$

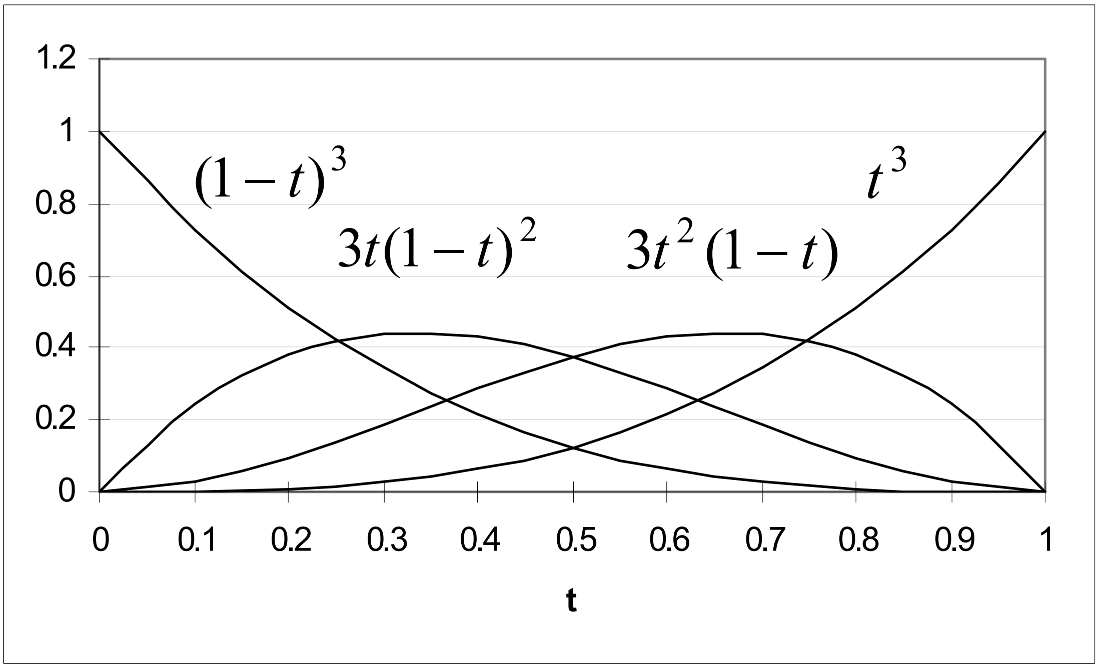


***Fig. 9.9** The cubic Bernstein polynomial curves*

Cubic Bernstein Polynomial

$$V = V_1(1-t)^3 + V_{c1}3t(1-t)^2 t + V_{c2}3t^2(1-t) + V_2 t^3$$

$$V = 1(1-t)^3 + 2.5\times 3t(1-t)^2 t - 2.5\times 3t^2(1-t) + 3t^3$$

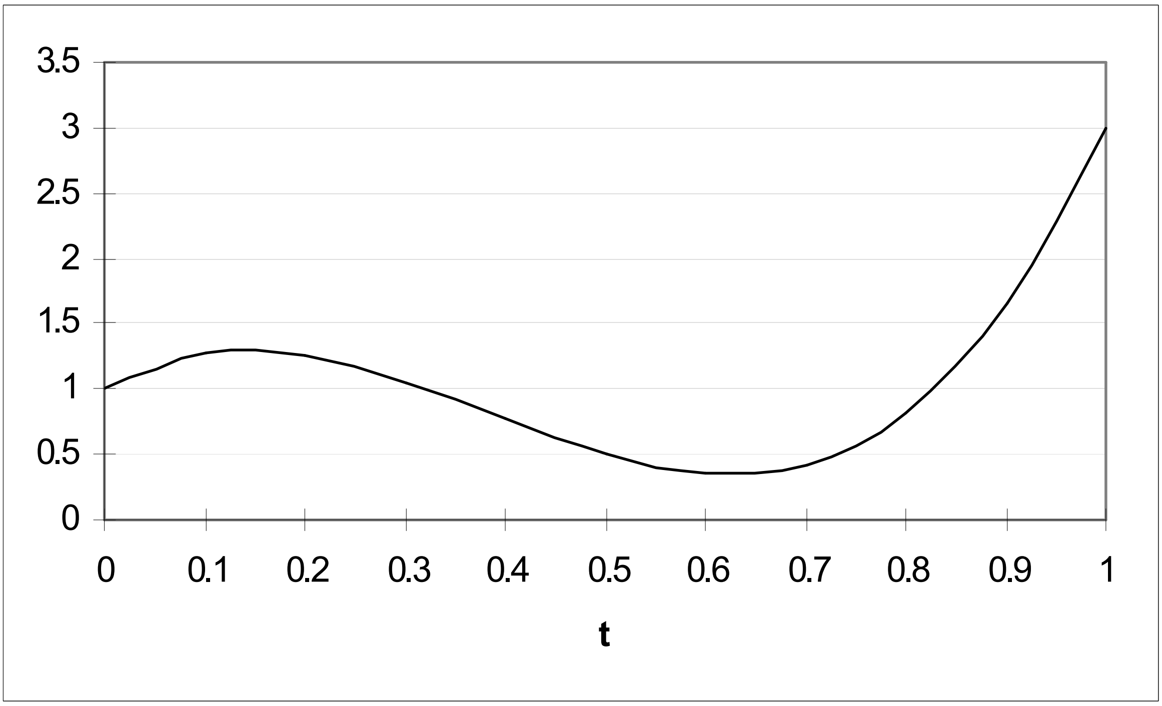


Fig. 9.10 *A cubic Bernstein polynomial through the values 1, 2.5, -2.5, 3.*

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Cubic Bézier Curve

The next step is to associate the blending polynomials with *x*- and *y*-coordinates.

$$x = x_1(1-t)^3 + x_{c1}3t(1-t)^2 t + x_{c2}3t^2(1-t) + x_2 t^3$$
$$y = y_1(1-t)^3 + y_{c1}3t(1-t)^2 t + y_{c2}3t^2(1-t) + y_2 t^3$$

$$(x_1, y_1) = (1, 1) \quad (x_{c1}, y_{c1}) = (2, 3)$$
$$(x_{c2}, y_{c2}) = (3, -2) \quad (x_2, y_2) = (4, 3)$$

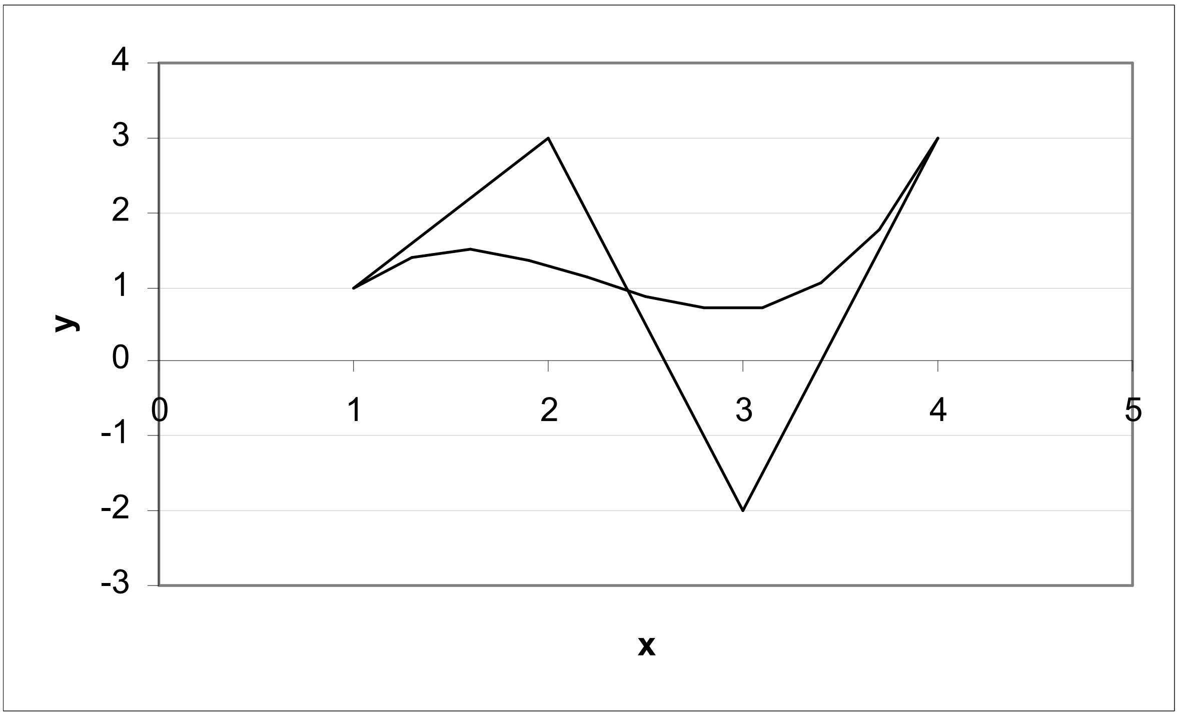

Fig. 9.11 A cubic B—zier curve.

Bézier curves using matrices

The quadratic Bernstein polynomial can be expanded to

$$(1-2t+t^2) \quad (2t-2t^2) \quad (t^2)$$

and can be written as the product of two matrices:

$$\begin{bmatrix} t^2 & t & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$V = V_1(1-t)^2 + 2t(1-t) + V_2 t^2$$

$$V = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_c \\ V_2 \end{bmatrix}$$

$$\mathbf{p}(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_c \\ \mathbf{P}_2 \end{bmatrix}$$

Bézier curves using matrices

The cubic Bézier curve can be represented by

$$\mathbf{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_{c1} \\ \mathbf{P}_{c2} \\ \mathbf{P}_2 \end{bmatrix}$$