

Matrices

Maths for Computer Graphics
2011-2012

Matrix

- Matrix an array of numbers arranged into m rows and n columns:

$$\begin{array}{c} \text{Rows,} \\ i = 1, \dots, m \end{array} \left\{ \begin{array}{ccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & a_{2n} \\ \vdots & & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{array} \right\}$$

Columns,
 $j = 1, \dots, n$

Row and Column Matrices (vectors)

- *row matrix* (or *row vector*) is a matrix with one row

$$r = \begin{pmatrix} r_1 & r_2 & r_3 & \cdots & r_n \end{pmatrix}$$

- *column vector* is a matrix with only one column

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

Square Matrix

- When the row and column dimensions of a matrix are equal ($m = n$) then the matrix is called square matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Matrix Equality

- Two $(m \times n)$ matrices **A** and **B** are equal if and only if each of their elements are equal. That is

A = **B** if and only if

$a_{ij} = b_{ij}$ for all $i = 1, \dots, m; j = 1, \dots, n$

Matrix Addition

2x2 Matrix addition

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

Matrix Addition

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & & b_{2n} \\ \vdots & & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} ;$$

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & & a_{2n} + b_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

Examples - Matrix Addition

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 4 & 3 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 3 & 2 & 1 \\ -4 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ -4 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 \\ -2 & 2 & 6 \\ 3 & 7 & 4 \end{bmatrix}$$

- The following matrix addition is ***not defined***

$$\begin{bmatrix} 1 & 2 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix} = ? \text{ (not defined!)}$$

Scalar – Matrix Multiplication

- Multiplication of a matrix $\mathbf{A}$ by a scalar is defined as follows

$$\alpha \mathbf{A} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & & \alpha a_{2n} \\ \vdots & & \ddots & \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{pmatrix}$$

- Example

$$\alpha = -2, \quad A = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 0 & 3 \end{pmatrix}, \quad \alpha A = \begin{pmatrix} -2 & -8 & -2 \\ -4 & 0 & -6 \end{pmatrix}$$

Matrix – Matrix Multiplication

- The product of two matrices **A** and **B** is defined only if the number of columns of **A** is equal to the number of rows of **B**.
- If **A** is ($m \times p$) and **B** is ($p \times n$), the product is an ($m \times n$) matrix **C**

$$\mathbf{C}_{m \times n} = \mathbf{A}_{m \times p} \mathbf{B}_{p \times n}$$

Matrix – Matrix Multiplication

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^p a_{ik} b_{kj}$$

Matrix – Matrix Multiplication

$$\mathbf{C} = \mathbf{AB} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & & a_{2p} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & & b_{2n} \\ \vdots & & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pn} \end{pmatrix}$$

$$\begin{pmatrix} a_{11}b_{11} + \cdots + a_{1p}b_{p1} & a_{11}b_{12} + \cdots + a_{1p}b_{p2} & \cdots & a_{11}b_{1n} + \cdots + a_{1p}b_{pn} \\ a_{21}b_{11} + \cdots + a_{2p}b_{p1} & a_{21}b_{12} + \cdots + a_{2p}b_{p2} & & a_{21}b_{1n} + \cdots + a_{2p}b_{pn} \\ \vdots & & \ddots & \vdots \\ a_{m1}b_{11} + \cdots + a_{mp}b_{p1} & a_{m1}b_{12} + \cdots + a_{mp}b_{p2} & \cdots & a_{m1}b_{1n} + \cdots + a_{mp}b_{pn} \end{pmatrix}$$

$$c_{ij} = \sum_{k=1}^p a_{ik} b_{kj}$$

Example - Matrix Multiplication

$$\mathbf{a} = (a_1, a_2, a_n) \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_n \end{pmatrix} \quad c_{1 \times 1} = a_{1 \times 3} b_{3 \times 1} \text{ (scalar)}$$

$$c = a \cdot b = (a_1, a_2, a_3) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Example - Matrix Multiplication

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \quad c_{nx1} = A_{n \times n} b_{n \times 1}$$

$$c = \mathbf{A} \mathbf{b} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_{11}b_1 + a_{12}b_2 + \cdots + a_{1n}b_n \\ a_{21}b_1 + a_{22}b_2 + \cdots + a_{2n}b_n \\ \vdots \\ a_{n1}b_1 + a_{n2}b_2 + \cdots + a_{nn}b_n \end{pmatrix}$$

Example - Matrix Multiplication

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \\ 5 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 7 \\ 6 & 8 \end{bmatrix}$$

$$C_{3 \times 2} = A_{3 \times 2} B_{2 \times 2}$$

$$C = A \cdot B = \begin{bmatrix} 3 & 0 \\ 1 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 12 & 21 \\ 10 & 15 \\ 32 & 51 \end{bmatrix}$$

Identity Matrix

- Identity Matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The identity matrix has the property that if A is a square matrix, then

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}$$