Reconstruction of Cosserat Rods

MSc Computer Animation and Visual Effects

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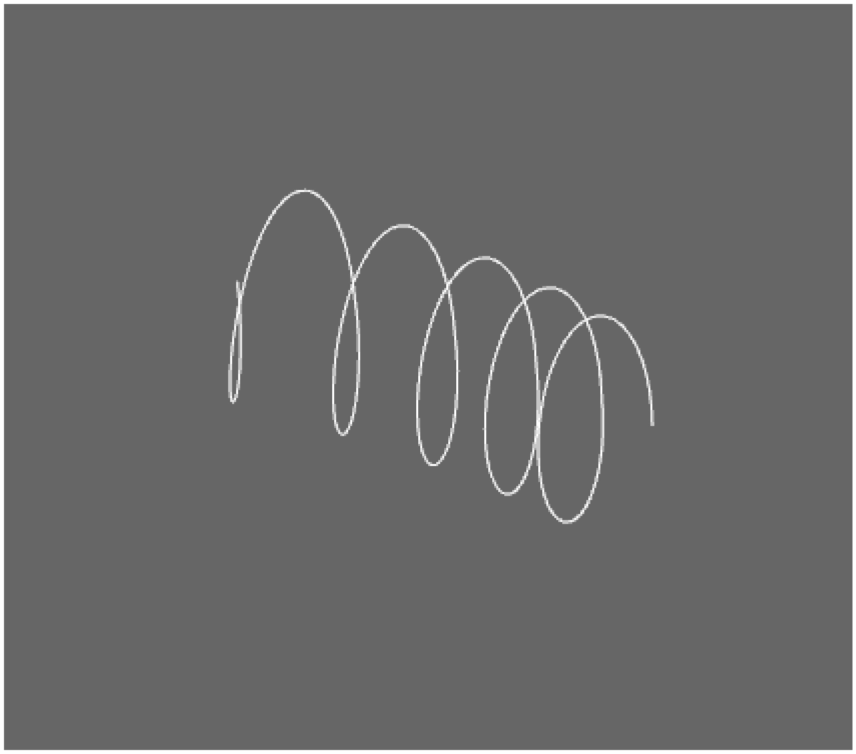
Introduction

Creating digital hair involves three main procedures: styling, simulation, and rendering. These tasks are challenging in that hair is a collective of thin, inextensible strands with complex and varying qualities. Unlike materials like fluid, hair is not as well-studied. However, similar materials like thread have been modeled, exhibiting behaviors similar to a strand of hair. These techniques are being adapted to simulate hair not only as single strands, but also its behavior as a collective. By examining the model of a strand of hair a thin, inextensible rod, it is possible to achieve more realistic simulations than alternative methods.

Previous Work

Early models of hair simulation focused on mass-spring systems, such as that proposed by Rosenblum (1991). These models treat hair as chains of hinges attached by stiff springs. The rigidity of the springs is to maintain the condition that a strand of hair cannot stretch, making it an inextensible material. While simple to understand and implement, the drawback of these the mass-spring method is that it cannot model twist and it is difficult to represent characteristics such as curl. Selle (2008) proposed a mass-spring system in which particles are linked by a system of tetrahedral springs, making it possible to create curled hairs.

A more thorough history of hair and strand simulations can be found in the survey conducted by Ward et al (2007) and and the course notes by Hadap (2007). Though they do not include systems developed in the time since 2007, the fundamentals of the most popular methods and their practical applications in the industry are well covered.



A Super-Helix with 5 segments

Dynamic Super Helices Model

For the notation of symbols in the equations, scalars will be represented as italics *a* and vectors with an overhead line $\overline{a}$. Primes indicate spatial derivatives $a' = \frac{\delta f}{\delta s}$.

Cosserat Rods

The cosserat model is a continuous model for a strand, which is an extremely thin material. Bergou (2008) demonstrated efficient ways of simulating such a material, as it compactly stores information about twist and bend.

Reconstruction

In the discretized model of Kirchhoff rods, the material curvatures for twist and bend represents its degrees of freedom. Since the rod $s \in [0, L]$ is broken down into N segments, the rod has 3N degrees of freedom. These segments S_Q where $Q\ (1 \le Q \le N)$ can be of different lengths, but for simplicity it is easier to make all segments the same length.

The generalized coordinates are stored in a 3N vector $\overline{q(t)}$, so that $q_{i,Q}(t)$ stores $k_1, k_2,$ *and* $k_0 = \tau$ and the material curvatures and twists can be explicitly expressed as

$$k_i(s,t) = \sum_{Q=1}^{N} q_{i,Q}(t)\,\chi_Q(t),$$

where $\chi_Q(t)$ = 1 if $s \in S_q$ and 0 otherwise (Bertails 2006).

From the generalized coordinates and the material frame, the strand at time t is reconstructed by integrating along $\overline{n_0}$ as it is the tangent to the centreline *r* such that:

$$\overline{r}'(s,t) = \overline{n_0}(s,t)$$

and $\overline{r}$' is the space derivative (Bertails, 2006). The beginning of the centreline $\overline{r_0}$ is clamped to the starting position of the strand, which would be the vertex from which it is grown. Since the Darboux vector $\overline{\Omega}$ is constant along each element as given by:

$$\overline{\Omega}' = \sum_i k_i \overline{n_i} + \overline{\Omega} \times \overline{\Omega} = 0$$

so we can use the Darboux vector of a segment Q in the reconstruction equations. For the equations we need $|\overline{\Omega}|$, the length of the Darboux vector, and $\overline{\varpi} = \overline{\Omega}/|\overline{\Omega}|$, which is the unit vector aligned with the Darboux vector. We also need the vectors $\overline{a}^{||}$ parallel and $\overline{a}^{\perp}$ perpendicular to $\overline{\varpi}$, given by:

$$\overline{a}^{||} = (\overline{a} \bullet \overline{\varpi})\overline{\varpi}$$
$$\overline{a}^{\perp} = \overline{a} - \overline{a}^{||}$$

For the integration, the material frame rotates around $\overline{\varpi}$ at the constant rate of Ω along the length of the curve. The angle is given by $\overline{\Omega}(s - s_Q^L)$. Then to reconstruct the material frames we use

$$\overline{n_i}(s) = \overline{n}_{i,L}^{Q||} + \overline{n}_{i,L}^{Q\perp} cos(\overline{\Omega}(s - s_Q^L)) + \overline{\varpi} \times \overline{n}_{i,L}^{Q\perp} sin(\overline{\Omega}(s - s_Q^L))$$

for i=0,1,2. Since the curve is continuous, $\overline{n}_{i,L}^{Q} = \overline{n}_i s_Q^L$, or the material frame at the end of that segment is equal to the material frame at the beginning of the next segment. Then we integrate over $\overline{n_0}(s)$ to get the centreline:

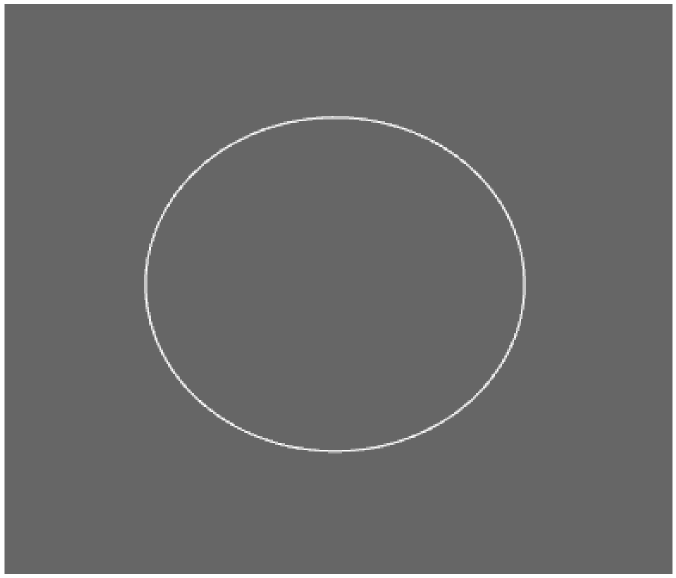
$$\overline{r}(s) = \overline{r}_L^{Q||}(s - s_Q^L) + \overline{n}_{0,L}^{Q\perp} \frac{sin(\overline{\Omega}(s - s_Q^L))}{|\overline{\Omega}|} + \overline{\varpi} \times \overline{n}_{0,L}^{Q\perp} \frac{1 - cos(\overline{\Omega}(s - s_Q^L))}{|\overline{\Omega}|}$$

Then the position of the centreline can be propogated as $\overline{r}_L^Q = \overline{r}(s_Q^L)$

Degenerate cases

For some values of τ, k^1, k^2, the hair behaves differently.

When $\tau = 0$, and $k^1 \neq 0 \; or \; k^2 \neq 0$, the rod forms an arc of a circle. When $k^1 = k^2 = 0$, the rod forms a straight line, which is twisted if $\tau \neq 0$ and untwisted if $\tau = 0$.



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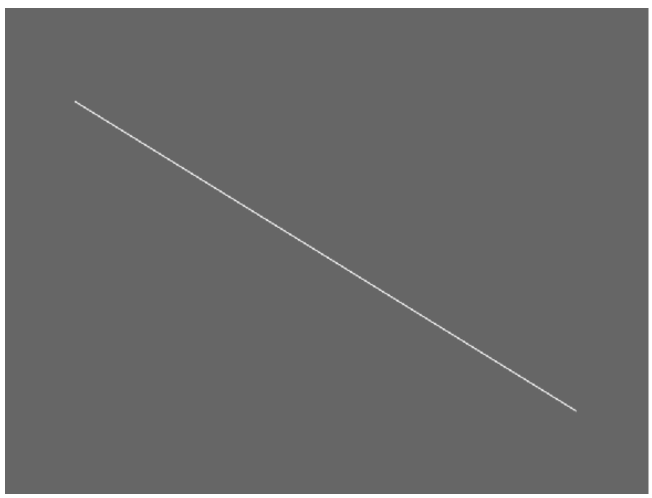

$k^1 = k^2 = 0$

Internal Energy

The dynamics of the super-helix are derived from Lagrangian equations of motion, which model kinetic energy, internal energy, and dissipation potential. In Bertails (2005), the configuration is calculated from the energies of the system. The potential energy of the rod is equal to:

$$E_{hair} = E_g + E_e$$

where E_g is the potential energy due to gravity and E_e is the internal elastic energy.

The eccentricity of the elliptical cross section of the hair is gives the moments of inertia for the equations. The eccentricity of the ellipse is given by

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

and

$$r = \frac{a+b}{2}$$

so that a is the major axis along n_1 and b is the minor axis along n_2 (Weisstein 2013b). Then it is solved for a and b using

$$a = \frac{2r}{1+\sqrt{1-e^2}}$$

$$b = 2r - a$$

Then the moments of inertia I_1 along n_1, I_2 along n_2, and the axial moment of inertia J are calculated using the moment of inertia for an ellipse (Weisstein 2013a).

Energy Minimization

The part I struggled with was finding the configuration of the generalized coordinates that achieved the lowest energy. The idea is to travel in the direction negative to the gradient of the energy equation, as the the gradient usually leads to a maximum. Some methods to achieve this might include following the gradient of steepest descent, testing with the Rosenbrock method, or the Davidon–Fletcher–Powell method (Bertails 2005).

The algorithm is basically (Bertails 2005):

1. Initialize the hair energy and generalized coordinates
2. Until the energy stops decreasing, keep computing the elastic energy, reconstructing the curve, calculating the potential energy from gravity, and minimize.

Lagrangian Equations of Motion

In Bertails (2005), the method described uses minimizing the potential and elastic energies of the strand to solve for the configuration of the generalized coordinates. Alternatively, other sources such as Bertails (2006) use Lagrangian motion to model forces, including gravity, twist, and collisions. I had originally planned to implement collisions using the penalty forces like in Bertail (2005).

Initialization

The most important initial values for each strand are determined by the clamped root position and the shape of the helix. The initial axis of the material frame, n_0, is the normalized tangent of the strand and given by the normal to the vertex from which the hair is grown. The helical radius gives the starting curvatures τ, k^1, k^2, given by equations:

$$\tau^n = \frac{\Delta h}{2\pi r^h}$$
$$k_1^n = \frac{1}{r^h}$$
$$k_2^n = 0$$

where r^h is the helical radius and Δh is the helical step.

From the normalized tangent n_0 the remaining axes of the material frame can be calculated. To get an orthonormal basis from the unit vector n_0, the method proposed by Hughes and Möller (1999) is used as it is simple and efficient. In this method, the smallest component of the vector is set to 0, while the other two components are swapped and the first is negated. This yields the vector n_1, so the last vector n_2 is calculated by taking the cross product $n_0 \times n_1$.

The default values I chose came from Bertails (2005) and Bertails (2006), which were taken from measurements of real clumps of hair. The papers give different values according to the type of hair desired, which range from thick and straight for smooth Asian hair, slightly curly for wavy and curly hair, or frizzy and tightly curled for African hair.
While incomplete, my original plan had also been to implement a styling tool for the super helix model that perhaps could make the unintuitive system easier to use. Since the natural curvatures of the hair define its initial resting shape, and the energy equations make use of the natural curvatures in calculating its potential energy, it would be simple to export the generalized coordinates as a way to save a hairstyle. This could also make it easier to save simulations as the generalized coordinates over each time step could be written out to a file and played back.

Results

The complication of the energy minimization step and trying to figure how to calculate the gradients and directions to solve for the minimum was where I ran into trouble. While the logic of finding the configuration of the curvatures that would yield the lowest energy made sense, calculating that configuration proved to be an entirely different matter. In Bertails (2005), the energy minimization step is barely mentioned, making it look deceivingly simple. I believe I set myself up for failure trying to tackle these equations without the mathematical background necessary. At the surface the equations seemed fair enough, but implementing them was much, much more difficult than I anticipated.

I had attempted to solve the equations by importing math libraries for solving the minimization step, but the trouble was more about being unable to derive the equations for the gradient. There are a multitude of math libraries, though I found little describing how to utilize them to solve for multidimensional systems. As the super helices model depends on generalized coordinates around local material frames, solving for directions of the gradient and steps to take were far more confusing. Perhaps the approach of using the Lagrangian equation for motion may have been more successful,.

The papers gloss over many of the crucial details assuming an audience well versed in physics, such as how to arrive at a configuration of the generalized coordinates from the dynamics equations. There is little said about time integration in any paper, making it unclear if the minimization was supposed to go until the minimum energy configuration was reached completely for a time step, or if each trial for finding the lowest energy might be a time step. The same system also switches between different methods of calculating the motion, going from energy minimization to Lagrangian motion. Perhaps this topic would have better been suited to an implementation in a technical computing program like MatLab. I spent entirely too long thinking I was so close to figuring it out instead of abandoning the topic. Instead of being stuck on this energy minimization step, I should have used the time to finish other components like the growing the hair from a mesh, interpolating strands, and create a user interface.

Conclusion

The super-helices model for hair simulation is a very unintuitive method, yet realistic results for hair simulation are known to have been achieved. The details of the simulation are complex and scattered through different papers, and some important parts do not seem to be easily studied without a background in complex mathematics. While deciphering the inconsistencies of the notation and equations makes implementation difficult, the most difficult part was switching from the mindset of particle dynamics to a discretized adaption of a continuous model.

Overall, the intimidating task of untangling these equations proved to be too much for me to take on in this project. Perhaps it would have been achievable if I had spent less time earlier in the project looking into implementations with OpenCL and trying to decide exactly what the project should focus on, and rather let those ideas develop as the project came along. However, this project of implementing the Cosserat rod model and its reconstruction as a discrete system is a solid foundation for continuing work with achieving simulated behavior, styling tools, and potential as a plugin for other packages.

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