Lecture 3 Geometry and Lines

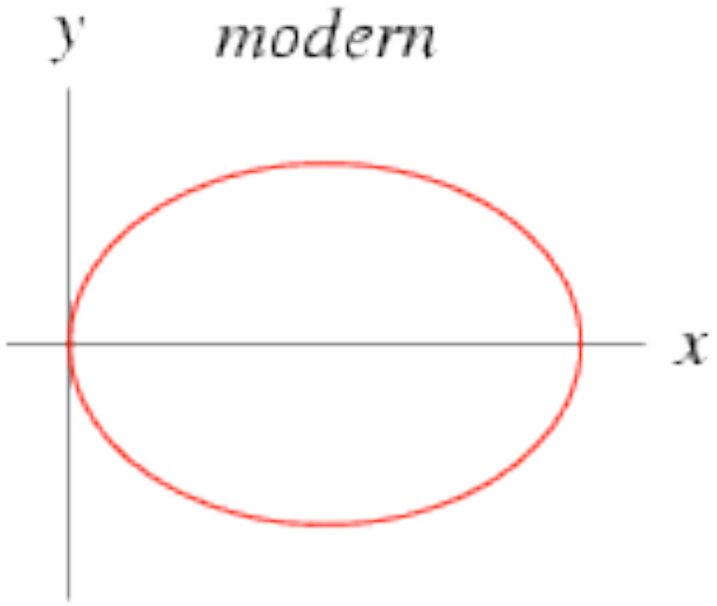
The Greek Alphabet

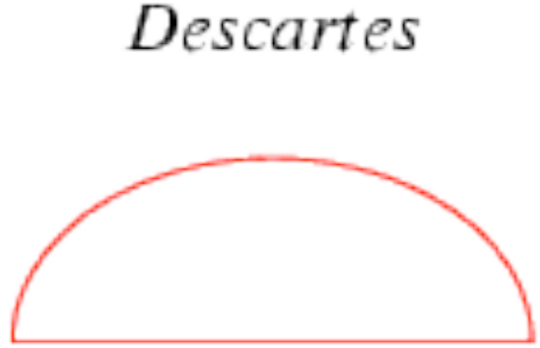
Aα Alpha	Bβ Beta
$\Gamma\gamma$ Gamma	$\Delta\delta$ Delta
E$\epsilon\varepsilon$ Epsilon	Zζ Zeta
Hη Eta	$\Theta\theta$ Theta
Iι Iota	Kκ Kappa
$\Lambda\lambda$ Lambda	Mμ Mu
Nν Nu	$\Xi\xi$ Xi
Oo Omicron	$\Pi\pi$ Pi
Pρ Rho	$\Sigma\sigma\varsigma$ Sigma
Tτ Tau	Yυ Upsilon
$\Phi\phi\varphi$ Phi	Xχ Chi
$\Psi\psi$ Psi	$\Omega\omega$ Omega

Cartesian Co-ordinates

- In 1636 Fermat (1601-1665) was working on a treatise titled "Ad locus planos et solidos isagoge" which outlined what we now call analytic geometry.
- Fermat never published his treatise, but shared his ideas with other mathematicians such as Blaise Pascal (1623-1662).
- In 1637 René Descartes (1596-1650) devised his own system of analytic geometry and published his results in the prestigious journal Géométrie.
- Ever since this publication Descartes has been associated with the xy-plane, which is why it is called the Cartesian plane.
- If Fermat had been more efficient with publishing his research results, the xy-plane could have been called the Fermatian plane!

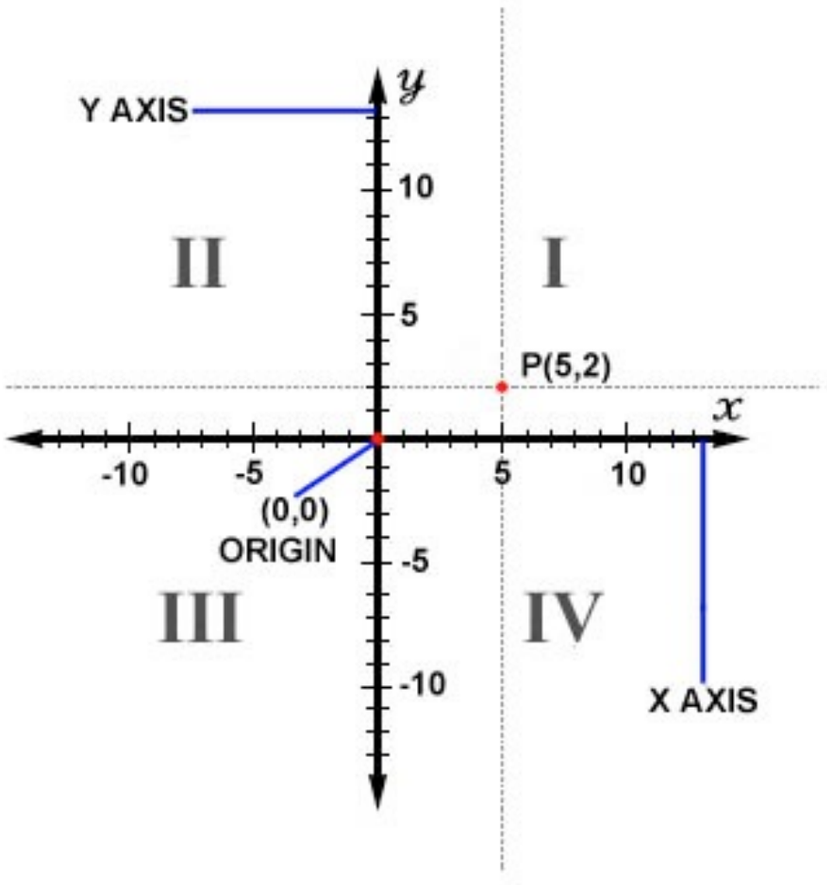
Cartesian Co-ordinates





- In René Descartes' original treatise the axes were omitted, and only positive values of the x- and the y- co-ordinates were considered, since they were defined as distances between points.
- For an ellipse this meant that, instead of the full picture which we would plot nowadays (left figure), Descartes drew only the upper half (right figure).

Cartesian Co-ordinates



2 DIMENSIONAL CARTESIAN COORDINATE SYSTEM

- The modern Cartesian co-ordinate system in two dimensions (also called a rectangular co-ordinate system) is commonly defined by two axes, at right angles to each other, forming a plane (an xy-plane).
- The horizontal axis is labelled x, and the vertical axis is labelled y.
- In a three dimensional co-ordinate system, another axis, normally labelled z, is added, providing a sense of a third dimension of space measurement.
- The axes are commonly defined as mutually orthogonal to each other (each at a right angle to the other).
- All the points in a Cartesian co-ordinate system taken together form a so-called Cartesian plane.
- The point of intersection, where the axes meet, is called the origin normally labelled O.
- With the origin labelled O, we can name the x axis O_x and the y axis O_y.
- The x and y axes define a plane that can be referred to as the xy plane. Given each axis, choose a unit length, and mark off each unit along the axis, forming a grid.

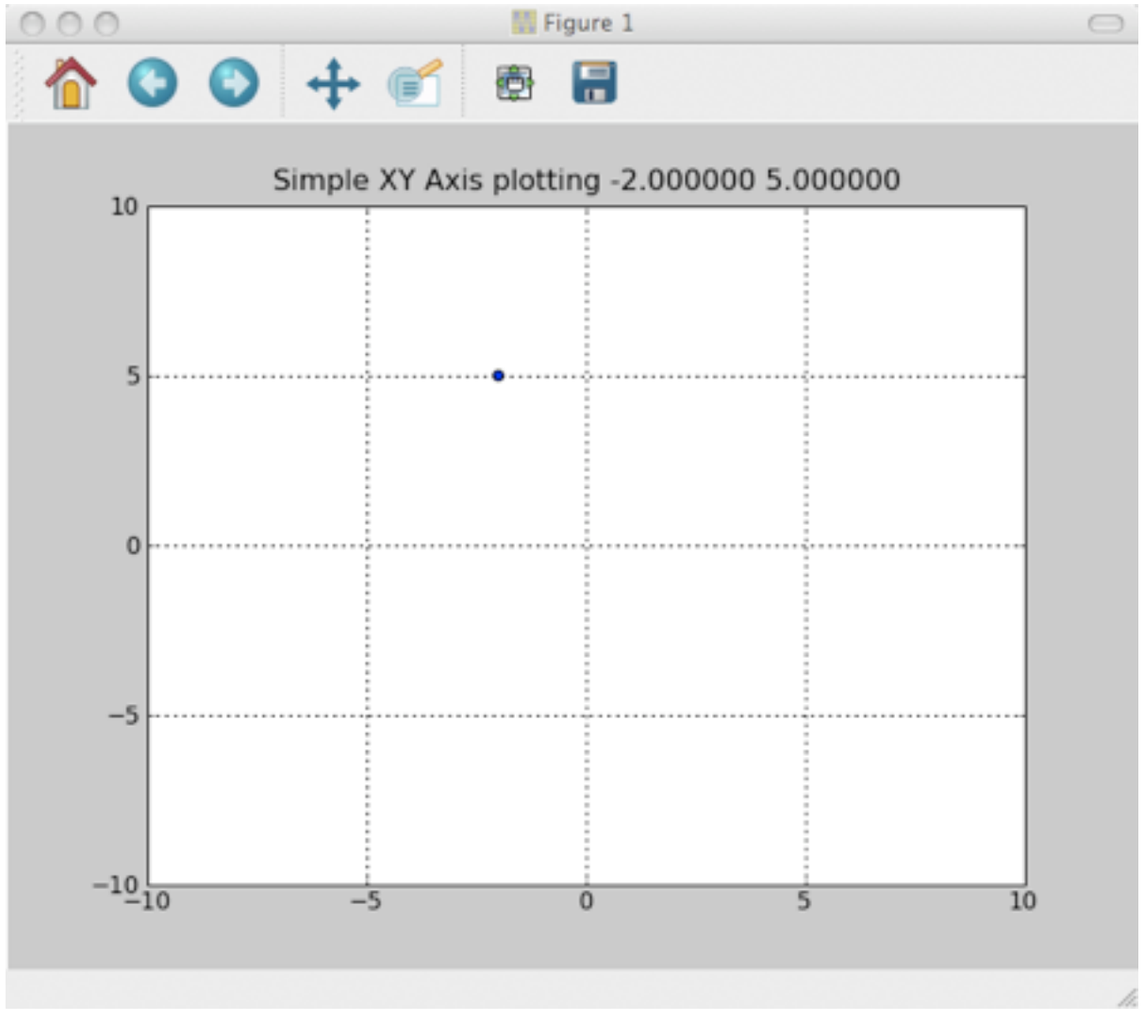
Cartesian Co-ordinates

- To specify a particular point on a two dimensional co ordinate system, you indicate the x unit first (abscissa), followed by the y unit (ordinate) in the form (x,y), an ordered pair.

- In three dimensions, a third z unit (applicate) is added, (x,y,z).

PlotPoint.py

```
1  #!/usr/bin/python
2
3  import matplotlib
4
5  matplotlib.use('Qt4Agg')
6
7
8  import matplotlib.pyplot as plt
9
10 x=float(raw_input("Enter a value for x > "))
11 y=float(raw_input("Enter a value for y > "))
12
13 # create a figure
14 fig = plt.figure()
15 # give it a title
16 plt.title("Simple XY Axis plotting %f %f"%(x,y))
17
18 # plot the point using the scatter function
19 plt.scatter([x],[y])
20
21 # set the range in the x
22 plt.xlim(-10,10)
23 # set the range in the y
24 plt.ylim(-10,10)
25
26 # draw a grid to show the axis
27 plt.grid(True)
28 plt.show()
```



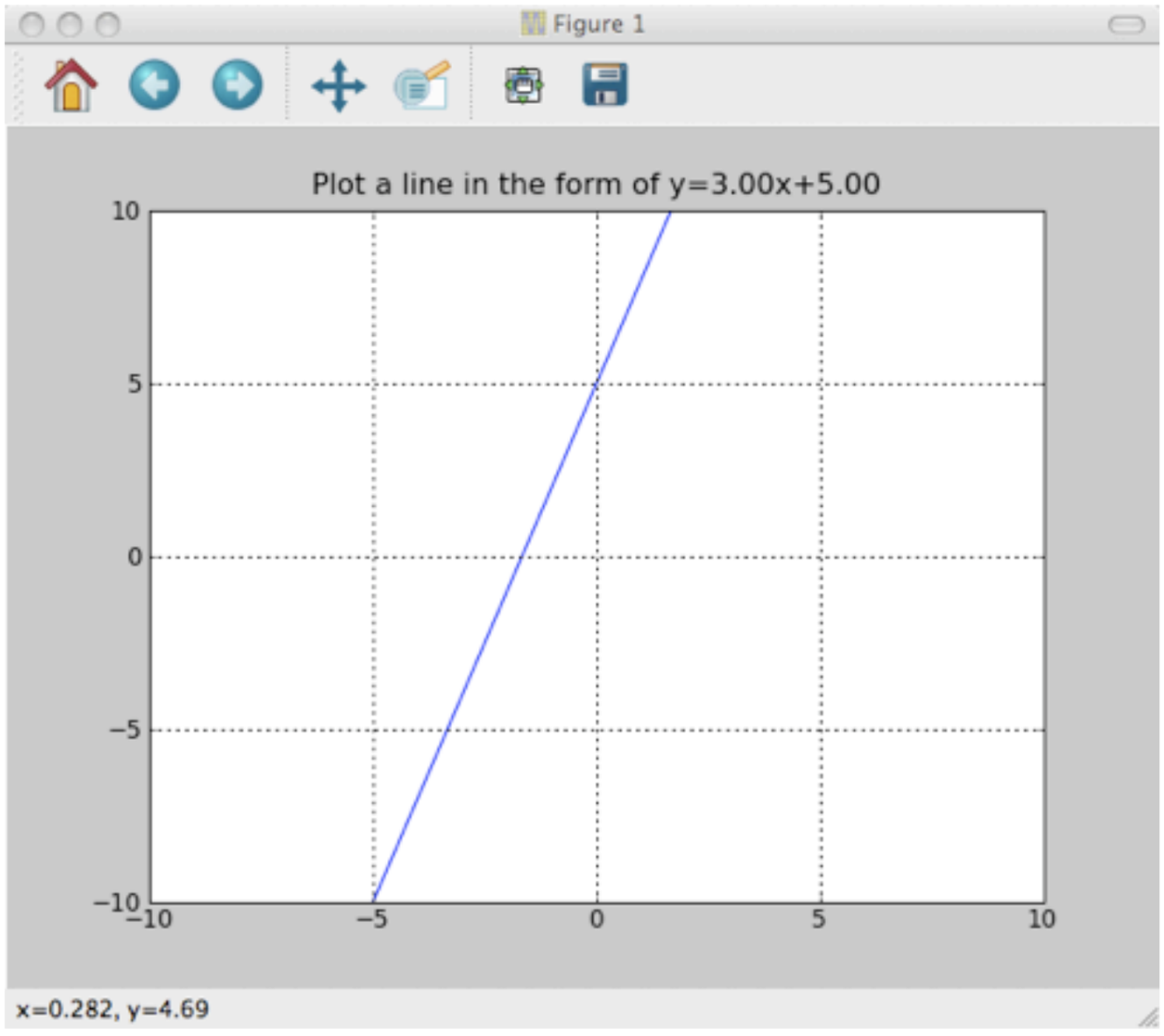
Straight Line Equation

- The simplest way to plot a straight line in Cartesian co-ordinates is using the "slope intercept" form

$$y = mx + b$$

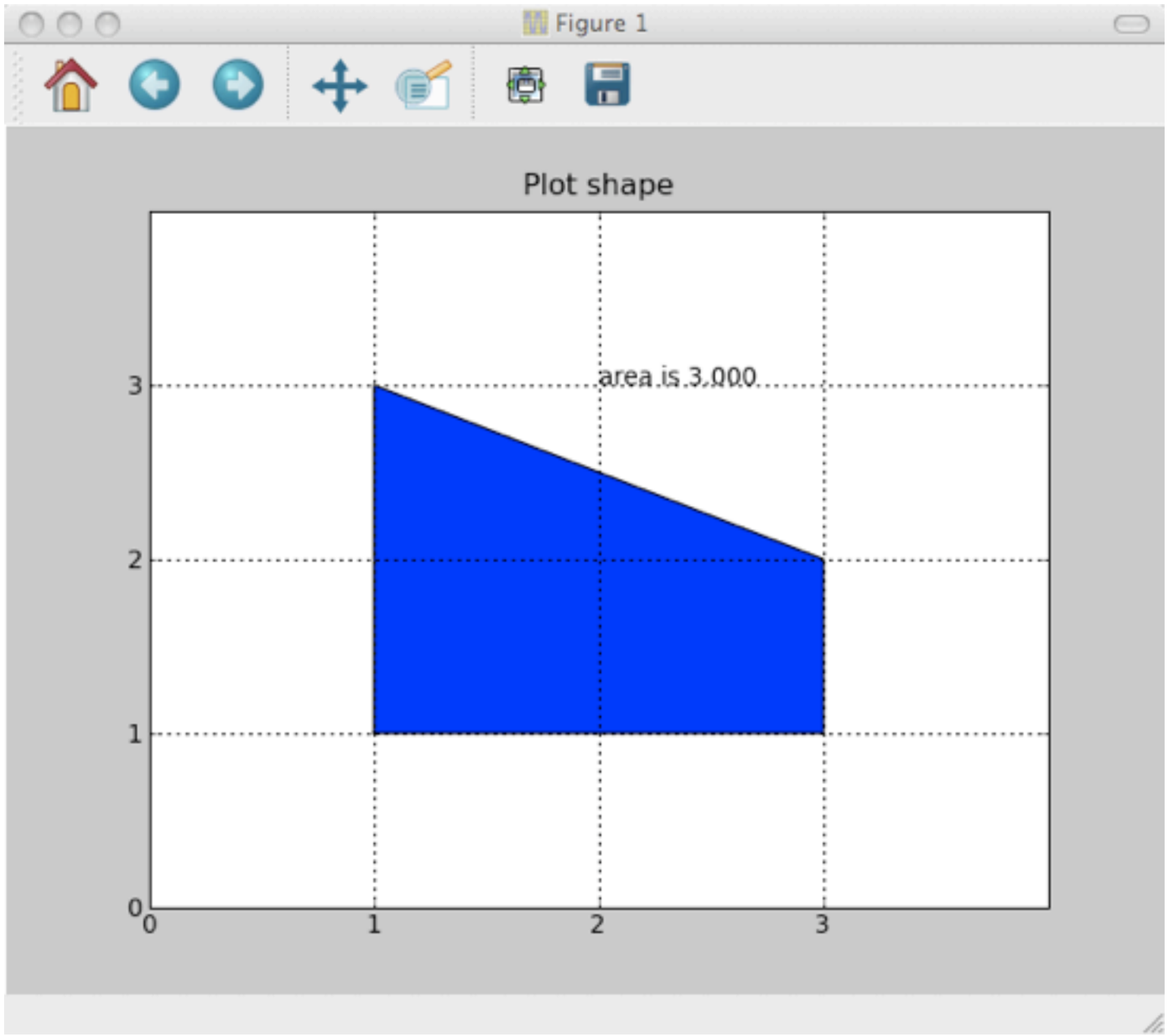
- This is called the slope-intercept form because "m" is the slope and "b" gives the y-intercept. (i.e. where it hits the y axis)

PlotLines.py




```
#!/usr/bin/python

import matplotlib

matplotlib.use('Qt4Agg')


import matplotlib.pyplot as plt
# get the values for m and b
print "Plot a line in the form of y=mx+b"
m=float(raw_input("Enter a value for m > "))
b=float(raw_input("Enter a value for b > "))


# create a figure
fig = plt.figure()
# give it a title
plt.title("Plot a line in the form of y=%0.2fx+%0.2f"%(m,b))

# create an empty list for the x and y values
x=[]
y=[]
# draw a grid to show the axis
for r in range(-10,10) :
  # make the x go from -10 to 10
  x.append(r)
  # calculate the line equation based on the range
  y.append(m*r+b)

plt.plot(x,y)
# set the range in the x
plt.xlim(-10,10)
# set the range in the y
plt.ylim(-10,10)
# draw the grid
plt.grid(True)

plt.show()
```

Geometric Shapes



X	Y
1	1
3	1
3	2
1	3

```
#!/usr/bin/python

import matplotlib
from random import *
matplotlib.use('Qt4Agg')


import matplotlib.pyplot as plt
# get the values for m and b


# create a figure
fig = plt.figure()
# give it a title
plt.title("Plot shape")

# create an empty list for the x and y values
x=[1,3,3,1,1]
y=[1,1,2,3,1]

#x=[randint(0,5),randint(0,5),randint(0,5),randint(0,5)]
#y=[randint(0,5),randint(0,5),randint(0,5),randint(0,5)]


# draw a grid to show the axis

plt.fill(x,y)
# set the range in the x
plt.xlim(0,4)
# set the range in the y
plt.ylim(0,4)

plt.xticks(range(0,4))
plt.yticks(range(0,4))

plt.axis([0,4,0,4])
# calculate the area of the polygon
area=0.5*((x[0]*y[1] - x[1]*y[0]) + (x[1]*y[2] - x[2]*y[1]) +
    (x[2]*y[3] - x[3]*y[2]) + (x[3]*y[0] - x[0]*y[3]))
plt.text(2,3,"area is %0.3f "%(area))

plt.grid(True)
plt.show()
```

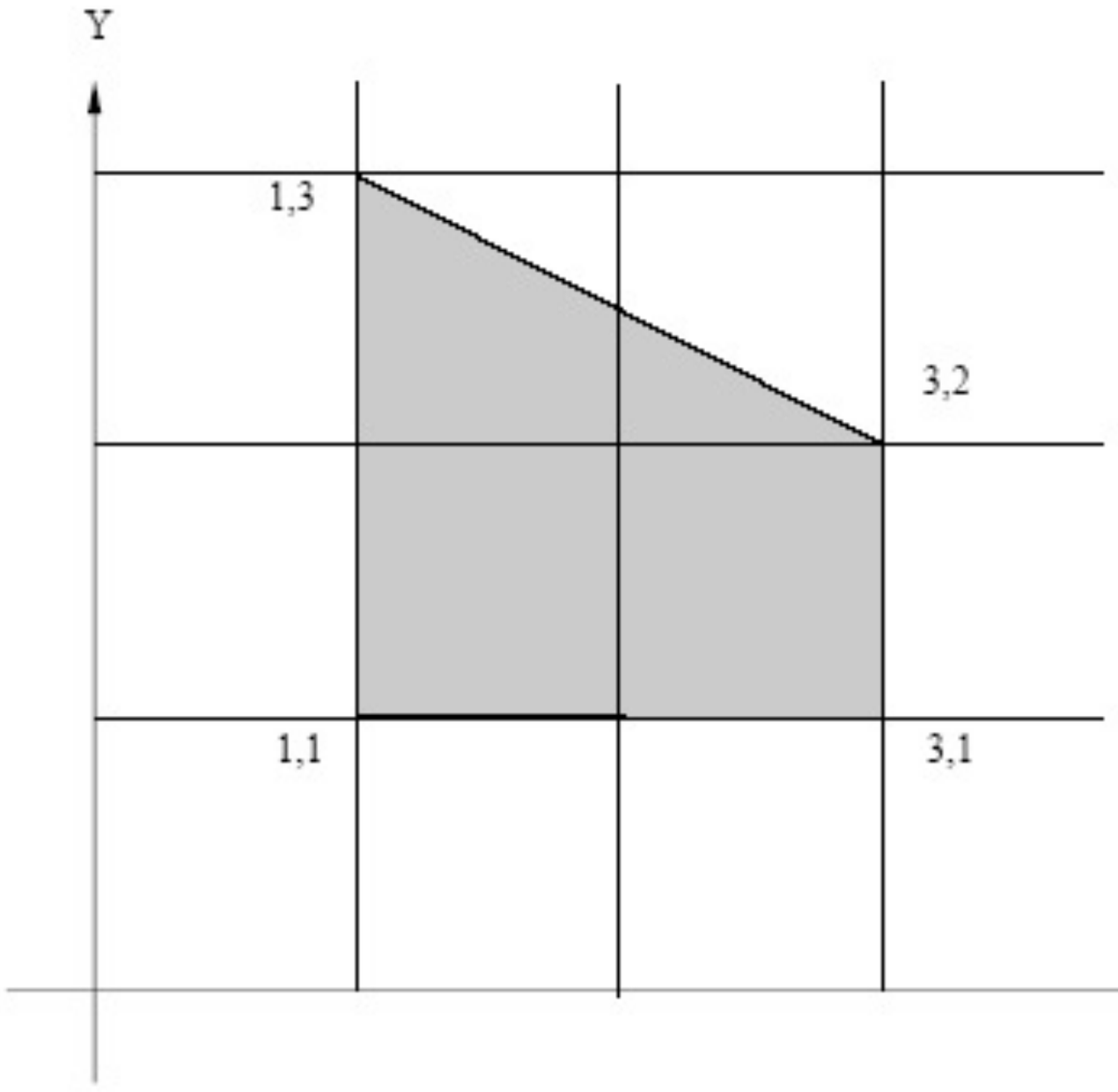
Areas of Shapes

- The area of a polygonal shape is computed by using the vertices by the following

x	y
x_0	y_0
x_1	y_1
x_2	y_2
x_3	y_3

$$\frac{1}{2}[(x_0y_1 - x_1y_0) + (x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_0 - x_0y_3)]$$

Geometric Shapes



x	y
1	1
3	1
3	2
1	3

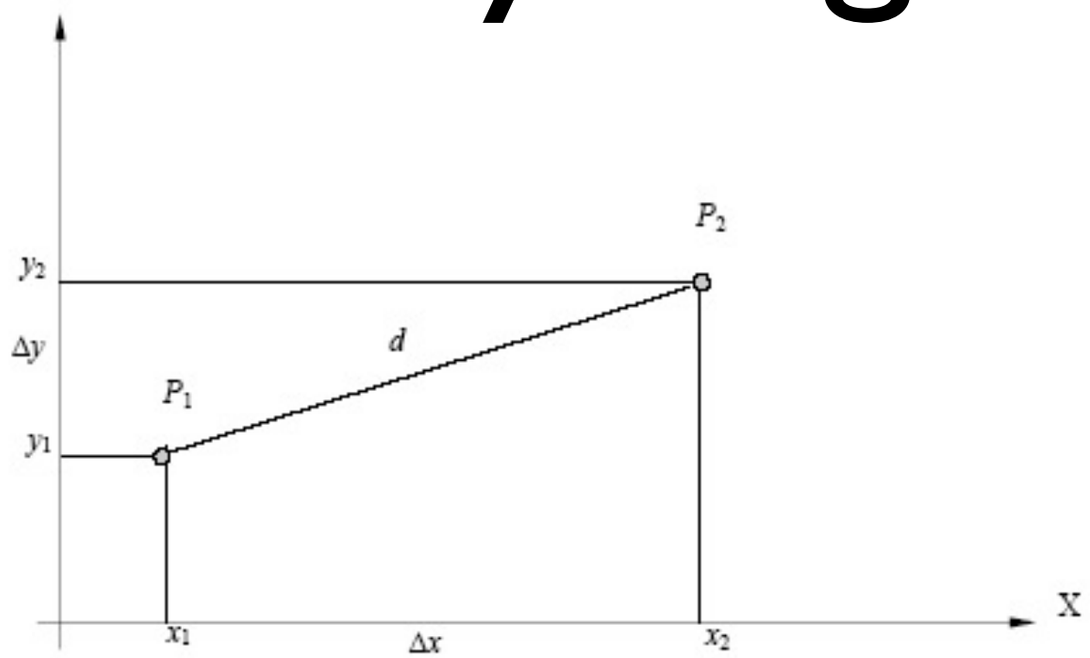
Reverse Order

x	y
1	1
1	3
3	2
3	1

$$\frac{1}{2}[(1 \times 3 - 1 \times 1) + (1 \times 2 - 3 \times 3) + (3 \times 1 - 3 \times 2) + (3 \times 1 - 1 \times 1)]$$

$$\frac{1}{2}[2 - 7 - 3 + 2] = -3$$

Theorem of Pythagorus in 2D



Given two arbitrary points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

We can calculate the distance between the two points

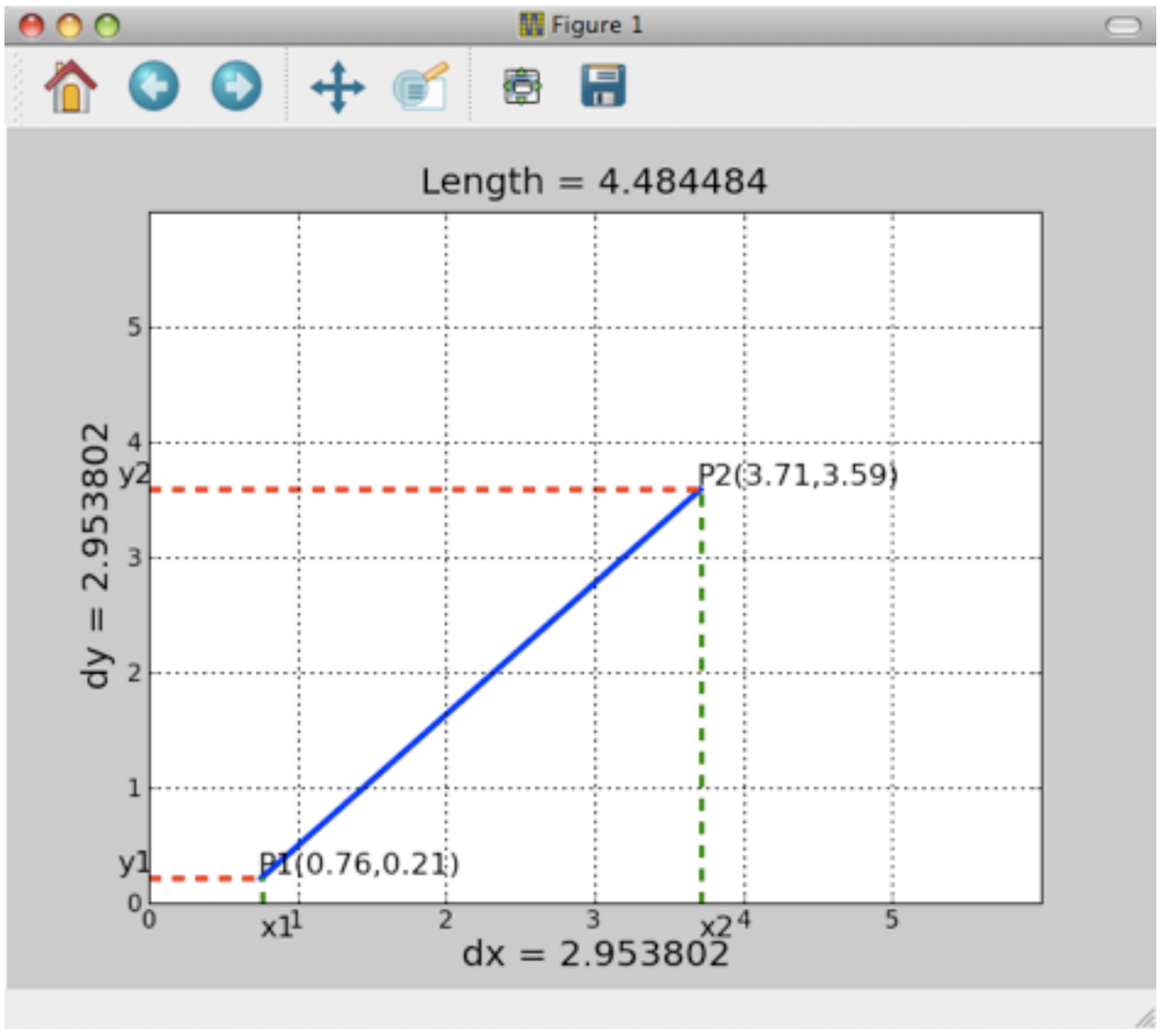
$\Delta x = x_2 - x_1$

$\Delta y = y_2 - y_1$

Therefore, the distance d between P_1 and P_2 is given by :

$d = \sqrt{\Delta x^2 + \Delta y^2}$

Pythagorous.py




```
1  # get two random points
2  def RandomPoint() :
3    x=[uniform(0,6),uniform(0,6)]
4    y=[uniform(0,6),uniform(0,6)]
5    return x,y
6
7
8  def Pythagorous(x,y) :
9    # draw a grid to show the axis
10   plt.cla()
11
12   # set the range in the x
13   plt.xlim(0,6)
14   # set the range in the y
15   plt.ylim(0,6)
16
17   plt.xticks(range(0,6))
18   plt.yticks(range(0,6))
19   plt.axis([0,6,0,6])
20
21
22   plt.grid(True)
23   plt.title("",fontsize=18)
24   # caluculate dX and dY
25   dX=x[1]-x[0]
26   dY=y[1]-y[0]
27   # get the length
28   length = math.sqrt(dX*dX+dY*dY)
29   # plot
30   plt.xlabel("dx␣=␣%f"%(dX),fontsize=18)
31   plt.ylabel("dy␣=␣%f"%(dX),fontsize=18)
32   plt.title("Length␣=␣%f"%(length),fontsize=18)
33   plt.plot(x,y,color='b',linewidth=3)
34   plt.plot([0,x[1]],[y[1],y[1]],'r--',linewidth=2)
35   plt.plot([0,x[0]],[y[0],y[0]],'r--',linewidth=2)
36
37   plt.plot([x[1],x[1]],[0,y[1]],'g--',linewidth=2)
38   plt.plot([x[0],x[0]],[0,y[0]],'g--',linewidth=2)
39   plt.text(x[0],y[0],"P1(%0.2f,%0.2f)" %(x[0],y[0]),fontsize=15)
40   plt.text(x[1],y[1],"P2(%0.2f,%0.2f)" %(x[1],y[1]),fontsize=15)
41
42   plt.text(-0.2,y[0],"y1" ,fontsize=15)
43   plt.text(-0.2,y[1],"y2" ,fontsize=15)
44
45   plt.text(x[0],-0.3,"x1" ,fontsize=15)
46   plt.text(x[1],-0.3,"x2" ,fontsize=15)
47   plt.show()
```

Euclid of Alexandria

- 300 BC to 275 B.C.
- The Father of Geometry
- Euclid's text Elements is the earliest known systematic discussion of geometry.

Axioms

- In mathematics, an axiom is any starting assumption from which other statements are logically derived.
- Euclidean geometry is an axiomatic system, in which all theorems ("true statements") are derived from a finite number of axioms.
- Near the beginning of the first book of the Elements, Euclid gives five postulates (axioms):

1. Any two points can be joined by a straight line.

2. Any straight line segment can be extended indefinitely in a straight line.

3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as centre.

4. All right angles are congruent.

5. Parallel postulate. If two lines intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

Lines

- A Line or Straight Line is usually defined as an (infinitely) thin, (infinitely) long, straight geometrical object
- Given two points, in Euclidean geometry, one can always find exactly one line that passes through the two points.
- The line provides the shortest connection between the points.
- Three or more points that lie on the same line are called **collinear**.
- Two different lines can intersect in at most one point;
- Two different planes can intersect in at most one line.

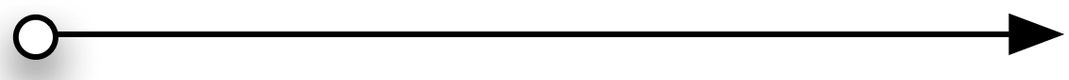
Lines

- A line is drawn as a straight line extending for some distance, with arrows on each end implying that the line continues beyond what is drawn.
- Rather than literally drawing a line, we can specify a line more precisely by describing it.
- In two-dimensional space, a line may be represented by the equation

$$ax + by + c = 0$$

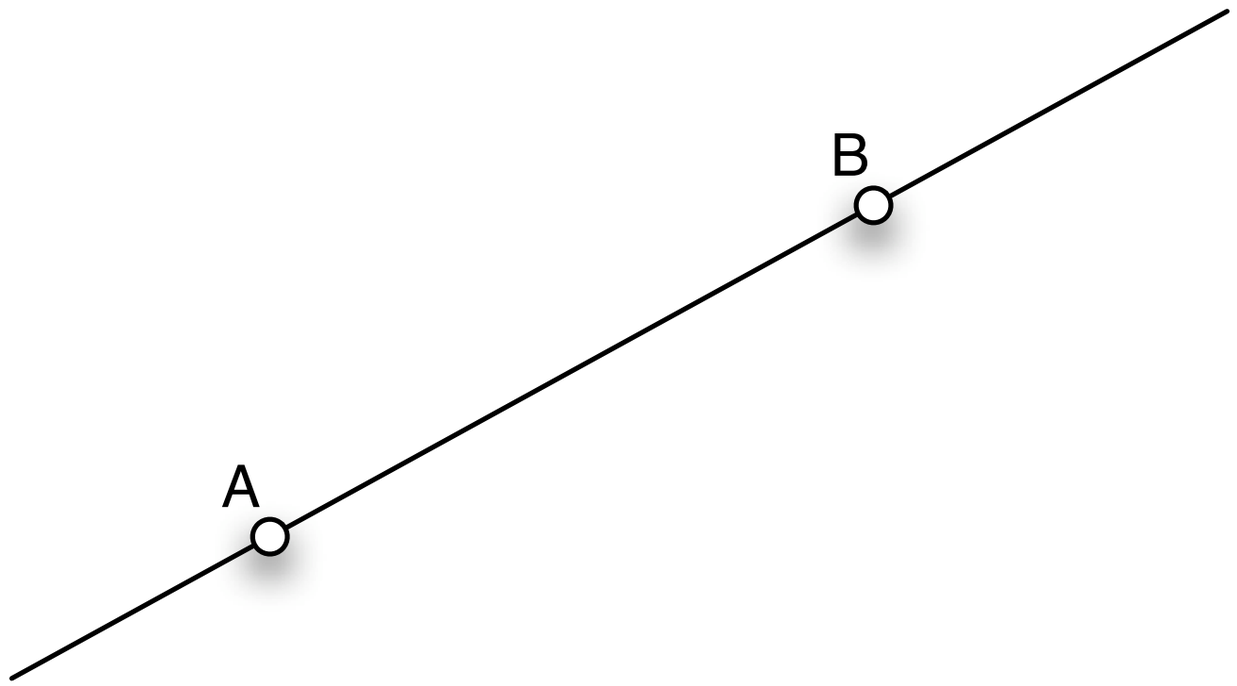
- Where a, b, and c are constants and x and y are variables.
- All points whose co-ordinates satisfy the equation will comprise the line.
- Another way of describing a line is by specifying any 2 points on the line.
- For example, "the line that goes through C and D".

Rays



- A ray can be thought of as "half" of a line.
- It starts at some point and extends infinitely far in one direction.
- Analytically we may define a ray with an equation defining a line plus a constraint.
- For example, a ray on the positive x-axis would be $y = 0, x \geq 0$
- Alternatively we may specify a starting point and an angle.
- Or we could specify two points, "the ray starting at C and going towards D".
- If you start at C heading towards D and continue walking in a straight line, all the points in your path are part of the ray, from C through to D and then beyond D to infinity ( )

Line Segments



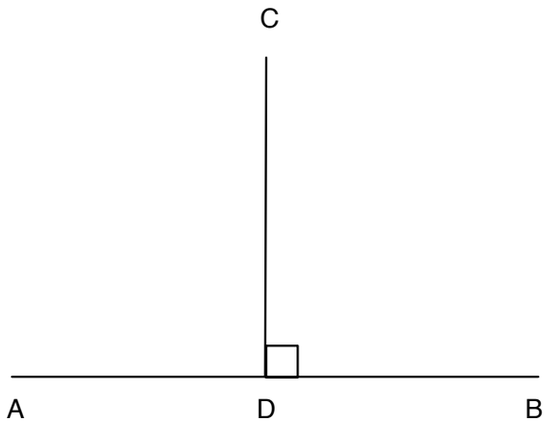
The Line passing between Points A and B is usually denoted as $\overleftrightarrow{AB}$

The finite line segment terminating at these points is denoted as $\overline{AB}$

Parallel Lines

- Two lines in two-dimensional Euclidean space are said to be parallel if they do not intersect.
- Two coplanar lines are said to be parallel if they never intersect.
- For any given point on the first line, its distance to the second line is equal to the distance between any other point on the first line and the second line.
- The common notation for parallel lines is $\|$ or sometimes $\backslash\backslash$
- If line m is parallel to line n, we write $m \parallel n$
- Lines in a plane either coincide, intersect in a point, or are parallel.
- Controversies surrounding the Parallel Postulate lead to the development of non-Euclidean geometries.

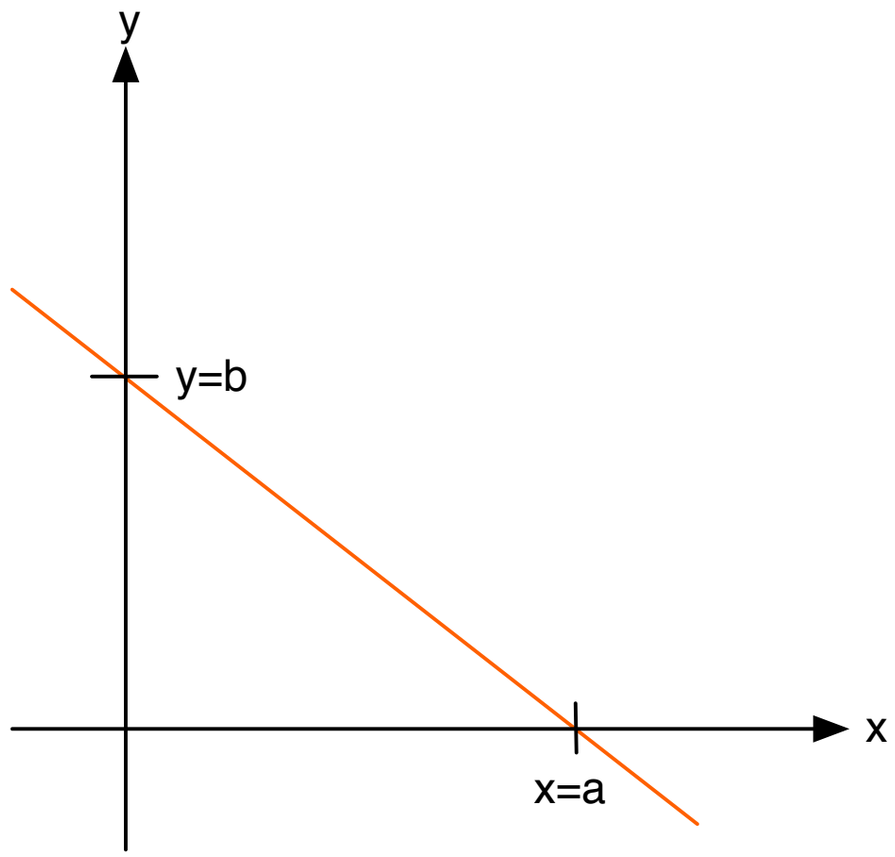
Perpendicular Lines



- Two lines, vectors, planes, etc, are said to be perpendicular if they meet at a right angle
- A right angle is an angle equal to half the angle from one end of a line segment to the other.
- A right angle is $\frac{\pi}{2}$ radians or 90^o
- Perpendicular objects are sometimes said to be "orthogonal."
- Mathematically Perpendicular Lines are represented as

$$\overline{AB} \perp \overline{CD}$$

Line Equations

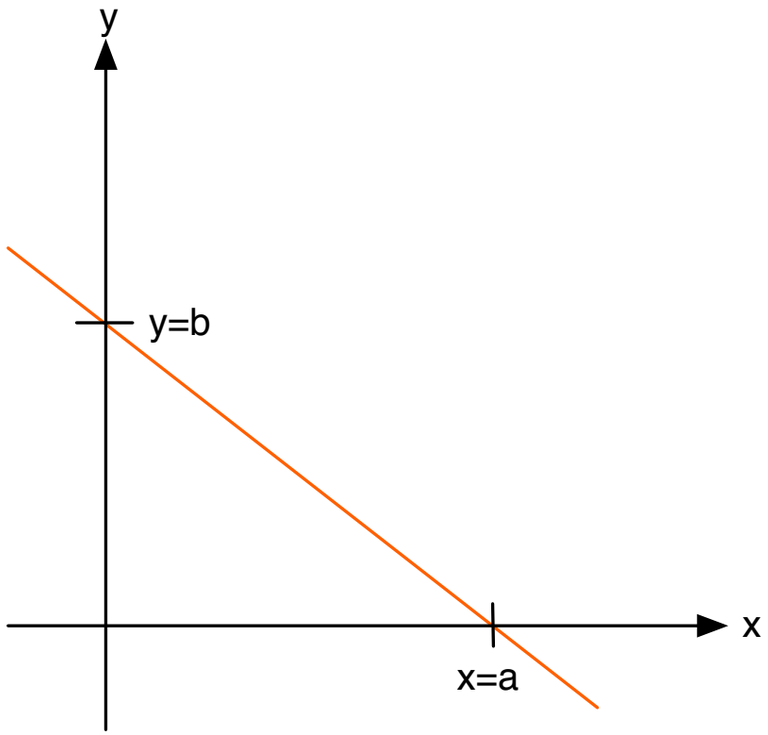


The line with x-intercept a and y-intercept b is given by the intercept form $\frac{x}{a} + \frac{y}{b} = 1$

The line through (x_1, y_1) with slope m is given in the point-slope form $y - y_1 = m(x - x_1)$

The line with y intercept b and slope m is given by the slope-intercept form $y = mx + b$

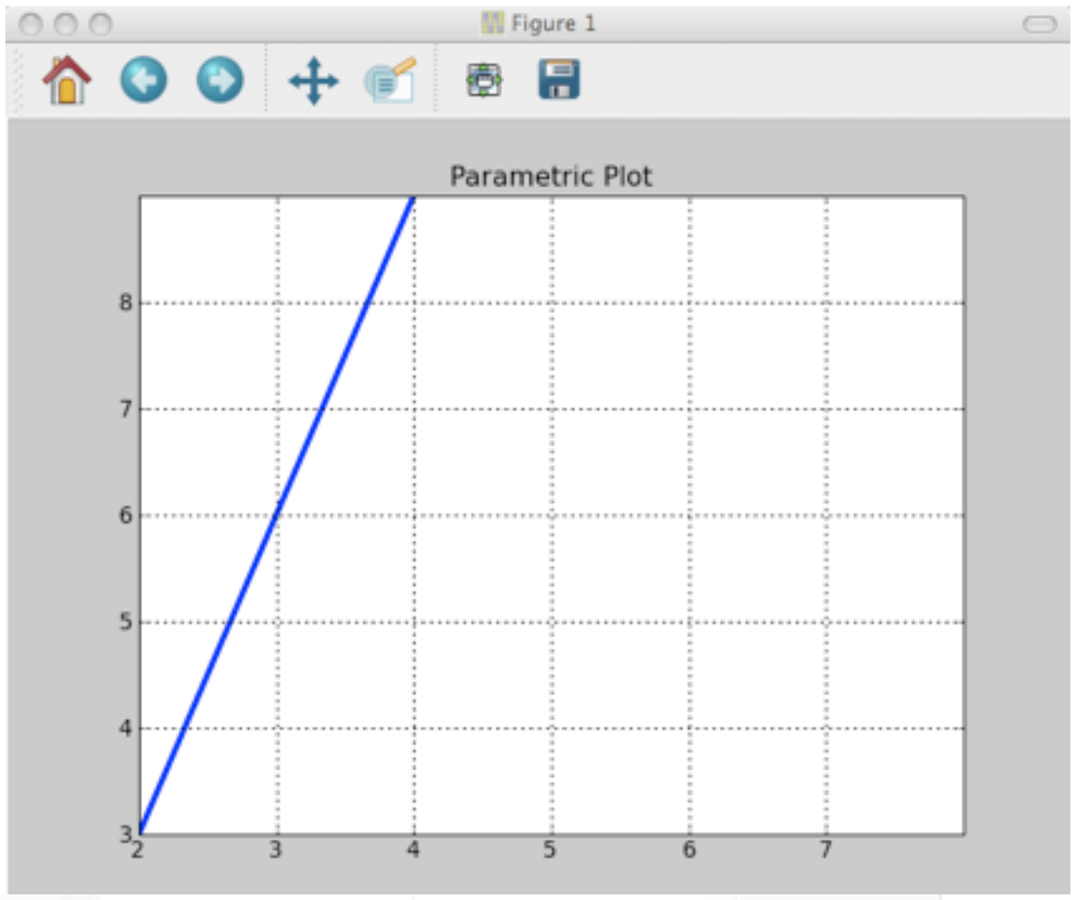
Line Equations



The line through (x_1, y_1) and (x_2, y_2) is given by the two point form

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

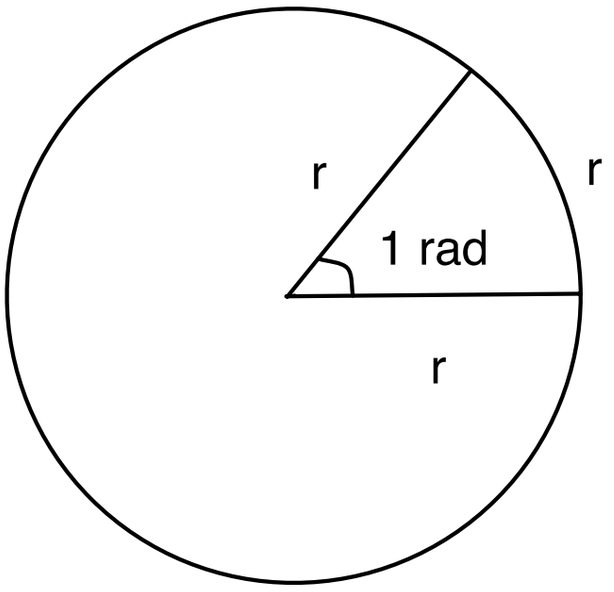
Parametric.py




```
[jmacey@neuromancer:Lecture3]$./Parametric.py
Please enter a value for x0  >2
Please enter a value for y0  >3
Please enter a value for a  >0.5
Please enter a value for b  >1.5
creating parametric data for t=0 t=6
x= [2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0]
y= [3.0, 4.5, 6.0, 7.5, 9.0, 10.5, 12.0]
```

```
1  #!/usr/bin/python
2
3  import matplotlib
4  from random import *
5  import math
6  matplotlib.use('Qt4Agg')
7
8  import matplotlib.pyplot as plt
9  # get the values for m and b
10
11 x0=int(raw_input("Please enter a value for x0  >"))
12 y0=int(raw_input("Please enter a value for y0  >"))
13
14 a=float(raw_input("Please enter a value for a  >"))
15 b=float(raw_input("Please enter a value for b  >"))
16
17 print "creating parametric data for t=0 t=6"
18 x=[]
19 y=[]
20 for t in range(0,7) :
21   x.append(x0+a*t)
22   y.append(y0+b*t)
23 print "x=",x
24 print "y=",y
25
26 # create a figure
27 fig = plt.figure()
28
29 # give it a title
30 plt.title("Parametric Plot")
31
32 # set the range in the x
33 plt.xlim(x0,x0+6)
34 # set the range in the y
35 plt.ylim(y0,y0+6)
36
37 plt.xticks(range(x0,x0+6))
38 plt.yticks(range(y0,y0+6))
39 plt.axis([x0,x0+6,y0,y0+6])
40
41 plt.grid(True)
42 plt.title("Parametric Plot ")
43
44 plt.plot(x,y,color='b',linewidth=3)
45 plt.show()
```

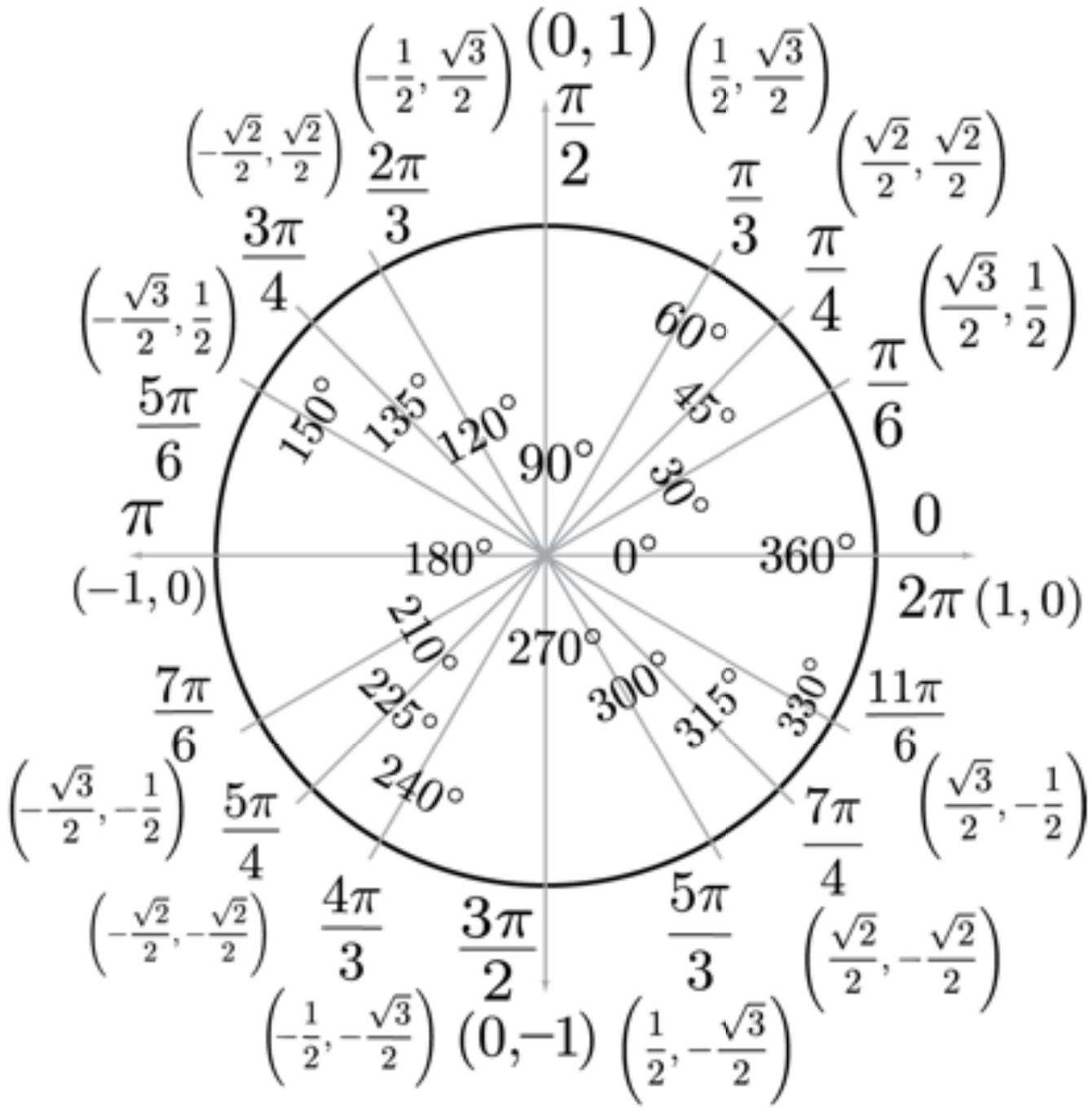
Degrees and Radians



The degree (or sexagesimal) unit of measure derives from defining one complete rotation as 360°

Each degree divides into 60 minutes, and each minute divides into 60 seconds.

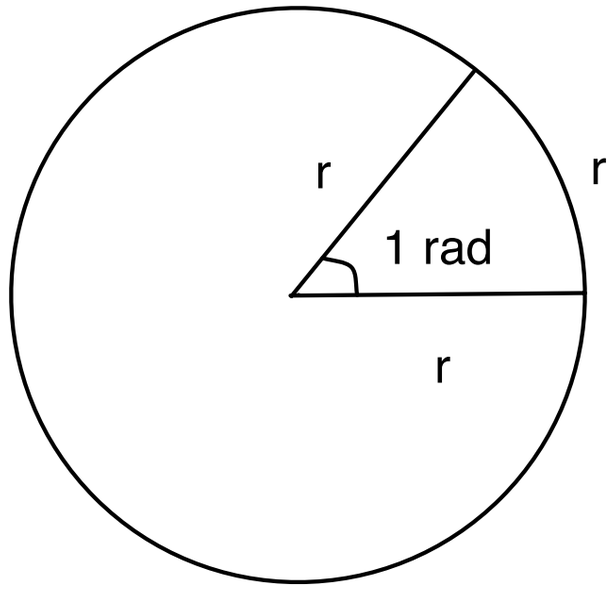
- The radian is the angle created by a circular arc whose length is equal to the circle's radius.
- The perimeter of a circle equals $2\pi r$,therefore 2π radians correspond to one complete rotation.
- 360° correspond to 2π radians, therefore 1 radian corresponds to $\frac{180^o}{\pi}$, approximately 57.3°



Notes

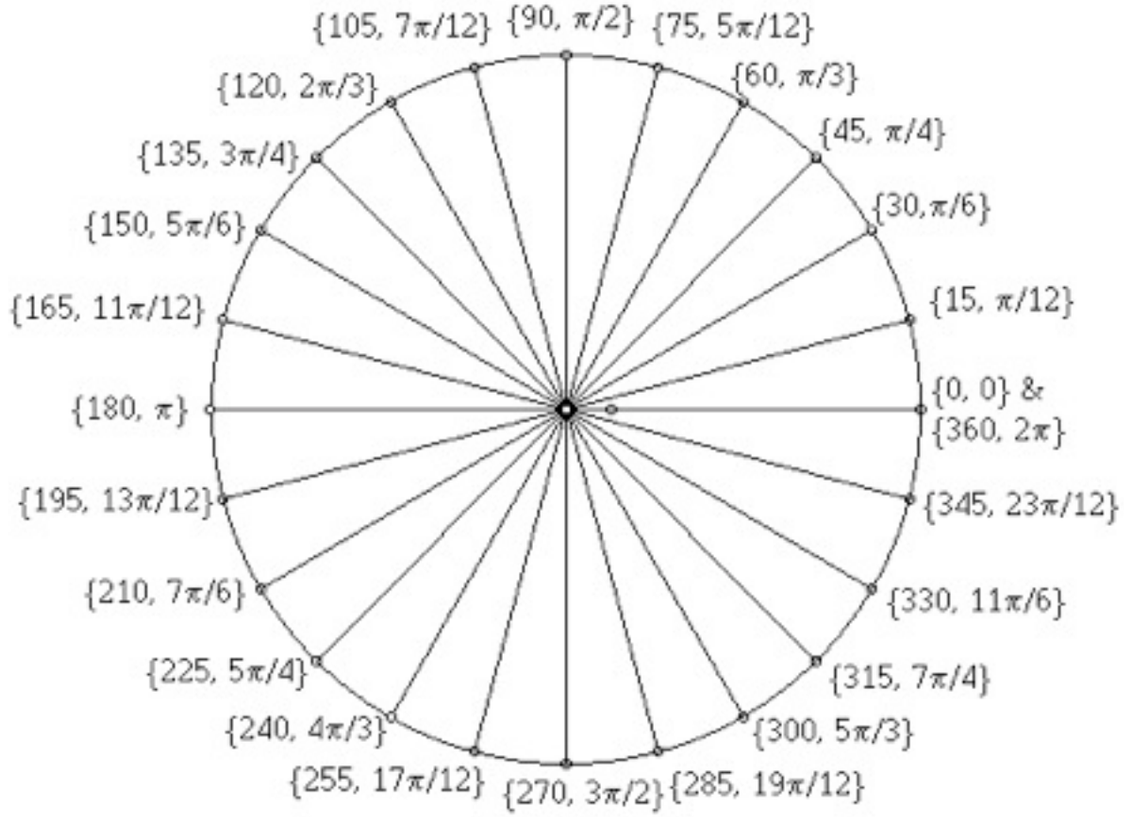
The unit circle should be second nature if you are working trigonometric function regardless of your level of math. You only need to memorize the special angles between 0 and π/2. The rest can be made by adding angles to make the rest of the chart. Memorizing those values can make a lot of classes like physics or engineering much easier.

Degrees and Radians



$$\frac{\pi}{2} = 90^o, \pi = 180^o, \frac{3\pi}{2} = 270^o, 2\pi = 360^o$$

Degrees to Radians Conversions



- As $2\pi = 360^o$ we can easily determine conversions from Degrees to Radians and back.
- This is important as most computer math functions require angles to be measured in radians

DegRad.py

```
1  #!/usr/bin/python
2  import math
3
4
5  for deg in range(0,370,20) :
6    print "degrees %3d = radians %f" %(deg,deg*(math.pi/180))
```

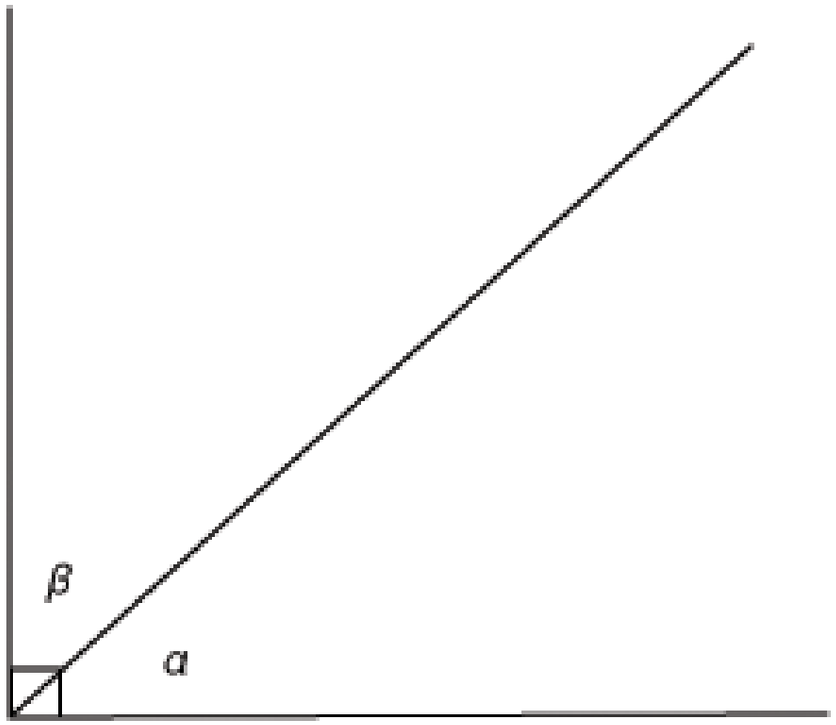
```
1  #!/usr/bin/python
2  import math
3
4  deg=180
5  rad=4.2352
6
7  print "%f in radians = "%(deg),math.radians(deg)
8  print "%f in degrees = "%(rad),math.degrees(rad)
```

```
[jmacey@neuromancer:Lecture3]$./DegRad2.py
180.000000 in radians =  3.14159265359
4.235200 in gegrees =  242.659085394
```

Angles

- An angle of $\frac{\pi}{2}$ radians or 90°, one-quarter of the full circle is called a right angle.
- Two line segments, rays, or lines (or any combination) which form a right angle are said to be either perpendicular or orthogonal
- Angles smaller than a right angle are called acute angles
- Angles larger than a right angle are called obtuse angles.
- Angles equal to two right angles are called straight angles.
- Angles larger than two right angles are called reflex angles.
- The difference between an acute angle and a right angle is termed the complement of the angle
- The difference between an angle and two right angles is termed the supplement of the angle.

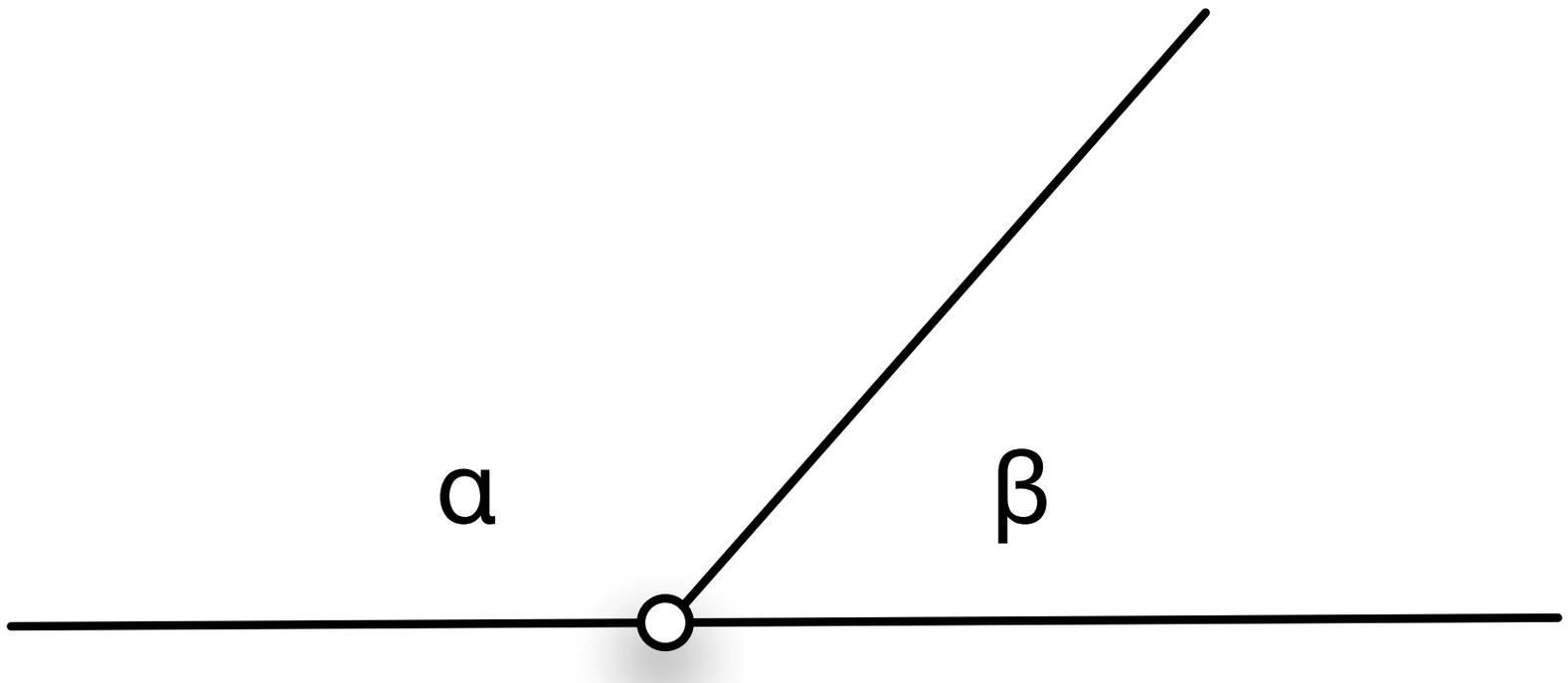
Complementary Angles



- Complimentary Angles sum to 90^o

$$\alpha + \beta = 90^0$$

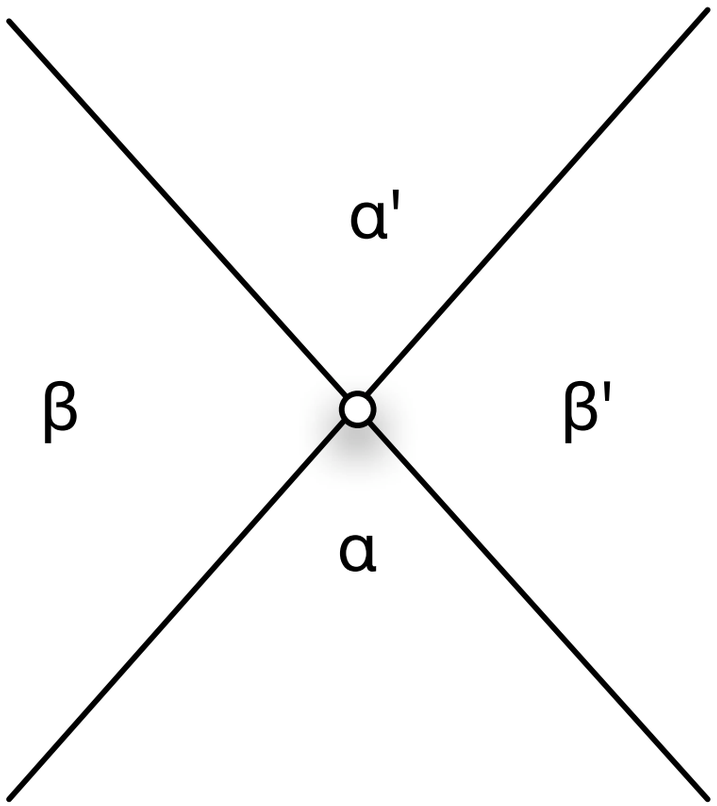
Supplementary Angles



- Supplementary angles sum to 180^0

$$\alpha + \beta = 180^0$$

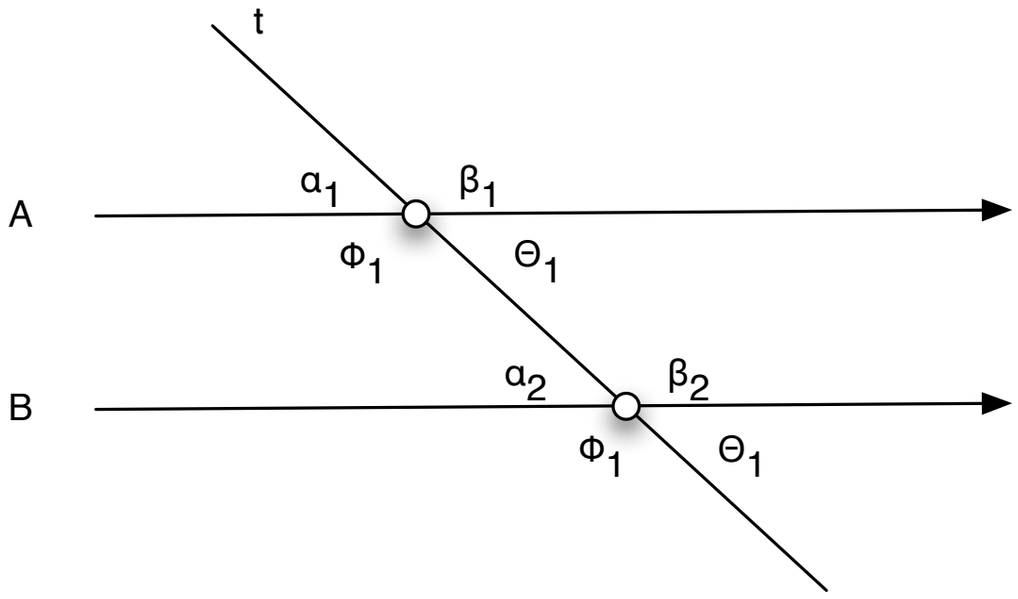
Vertical Angles



- Two pairs of vertical angles are created by two intersecting straight lines

$$\alpha = \alpha' \text{ and } \beta = \beta'$$

Transversal Line

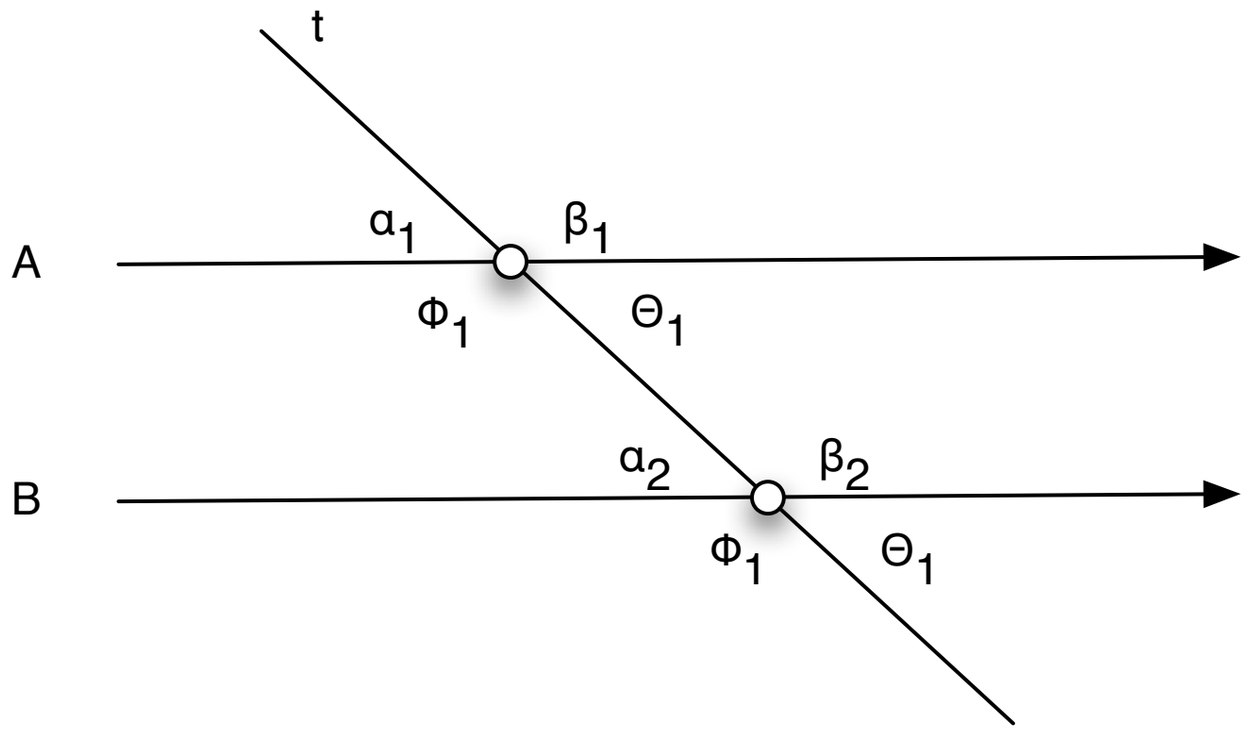


- A transversal line is a line that passes through two or more other coplanar lines at different points, especially when the other coplanar lines are also parallel.

If A is parallel to B, and transversal t crosses A with angle α

then it must also cross B with angle θ, such that the sum of all internal angles is 180^o

Transversal Line



interior angles $\alpha_2, \beta_2, \theta_1, \phi_1$

alternate interior angles $\theta_1 = \alpha_2$, $\phi_1 = \beta_2$

corresponding angles $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\phi_1 = \phi_2$, $\theta_1 = \theta_2$

opposite angles $\alpha_1 = \theta_1$, $\phi_1 = \beta_1$, $\alpha_2 = \theta_2$, $\phi_2 = \beta_2$

exterior angles $\alpha_1, \beta_1, \theta_2, \phi_2$

alternate exterior angles $\alpha_1 = \theta_2$, $\beta_1 = \phi_2$

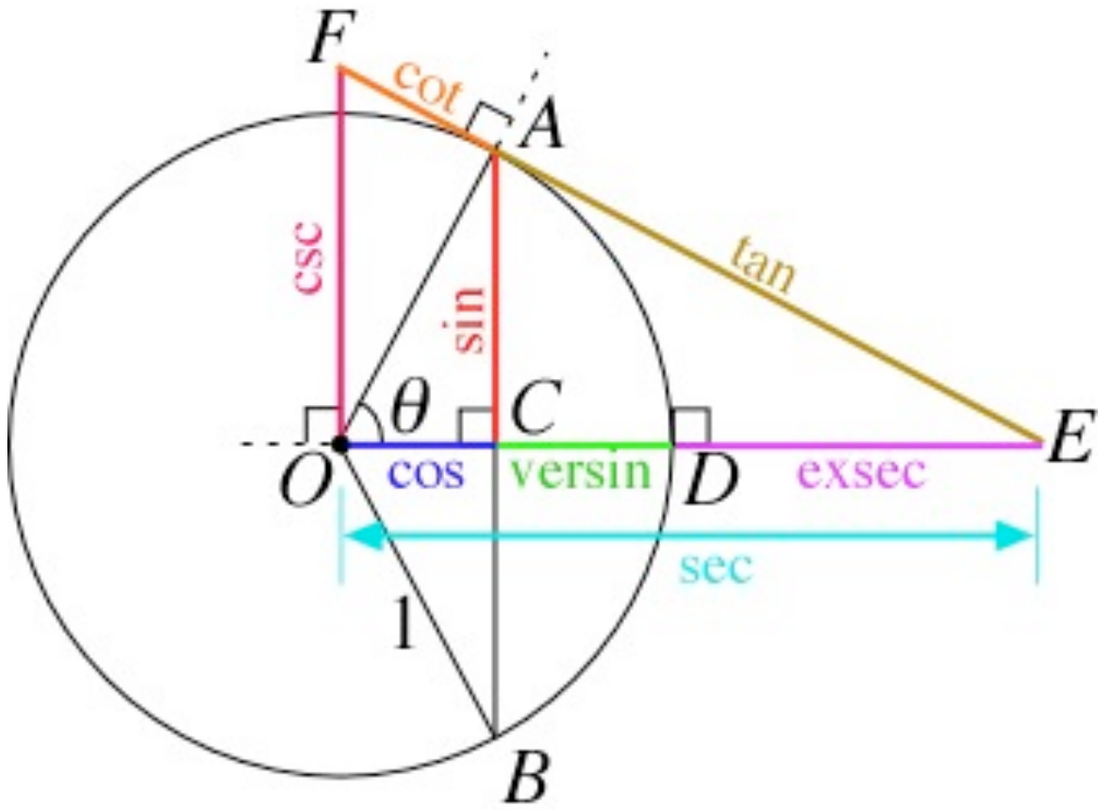
The trigonometric ratios

- The Hindu word ardha-jya meaning "half-chord" was abbreviated to jya ("chord"), which was translated by the Arabs into jiba, and corrupted to jb.
- Other translators converted this to jaib, meaning "cove","bulge" or "bay", which in Latin is sinus.
- Today, the trigonometric ratios are known as sin, cos, tan, cosec, sec and cot.

The Six Trigonometric Functions

Function	Abbreviation	Relation
Sine	sin	$\sin\theta = \cos(\frac{\pi}{2} - \theta)$
Cosine	cos	$\cos\theta = \sin(\frac{\pi}{2} - \theta)$
Tangent	tan	$\tan\theta = \frac{1}{\cot\theta} = \frac{\sin\theta}{\cos\theta} = \cot(\frac{\pi}{2} - \theta)$
Cotangent	cot	$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \tan(\frac{\pi}{2} - \theta)$
Secant	sec	$\sec\theta = \frac{1}{\cos\theta} = \csc(\frac{\pi}{2} - \theta)$
Cosecant	csc (or cosec)	$\csc\theta = \frac{1}{\sin\theta} = \sec(\frac{\pi}{2} - \theta)$

Alternative Representation of Trigonometric Ratios

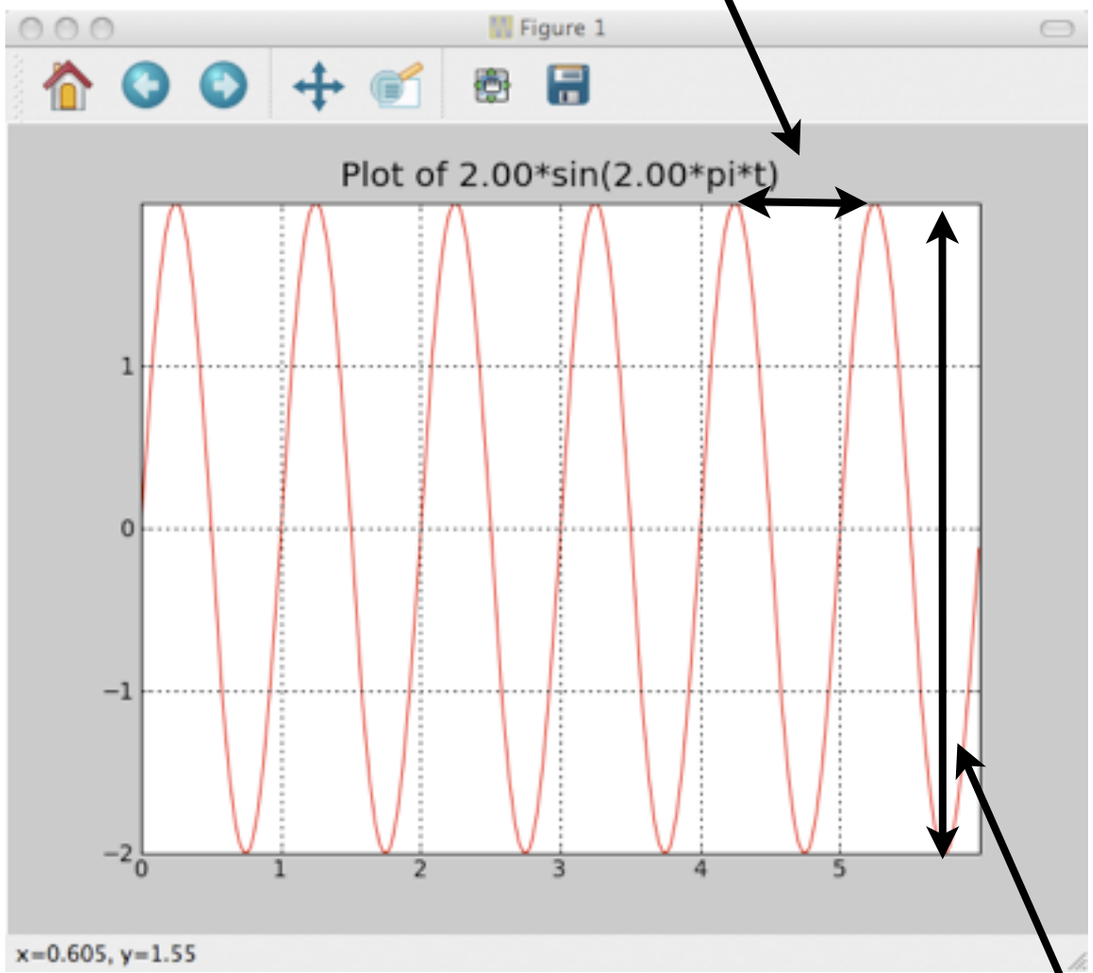


All of the basic trigonometric functions can be defined in terms of a unit circle centred at O

In particular, for a chord AB of the circle, where θ is half of the subtended angle
$\sin(\theta)$ is AC (half of the chord)
$\cos(\theta)$ is the horizontal distance OC
The other functions are also shown for clarity.

Sine.py

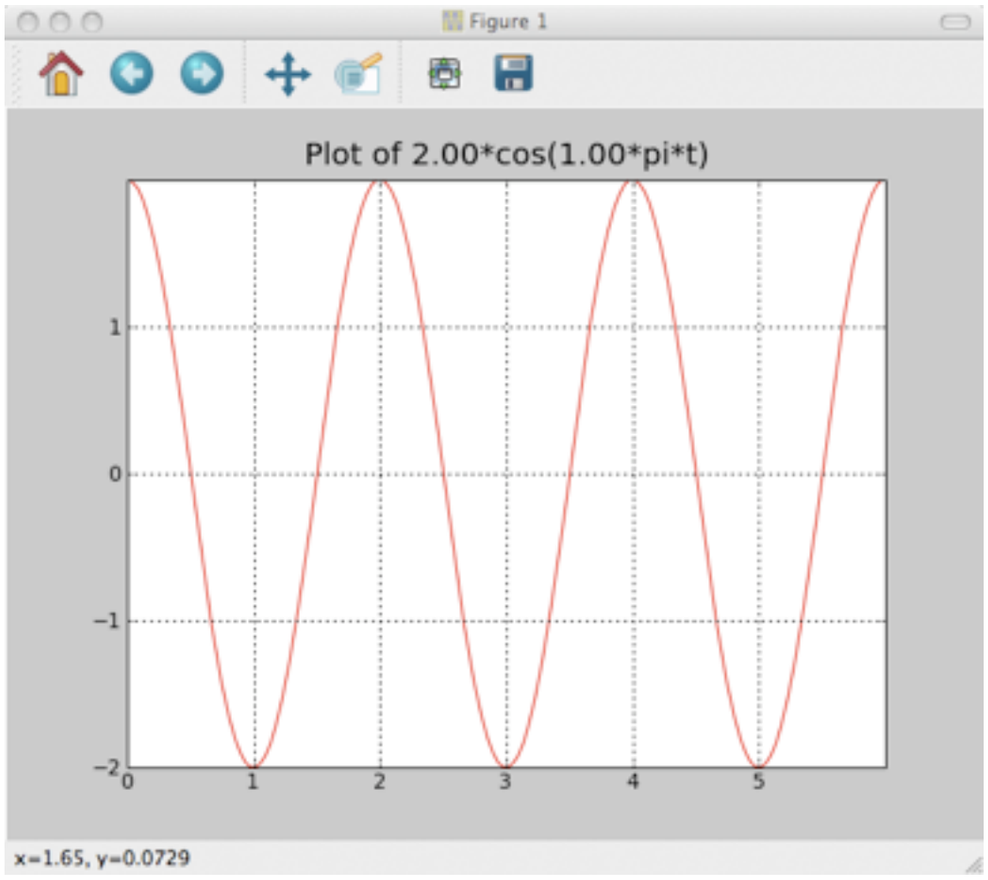
Frequency



Amplitude

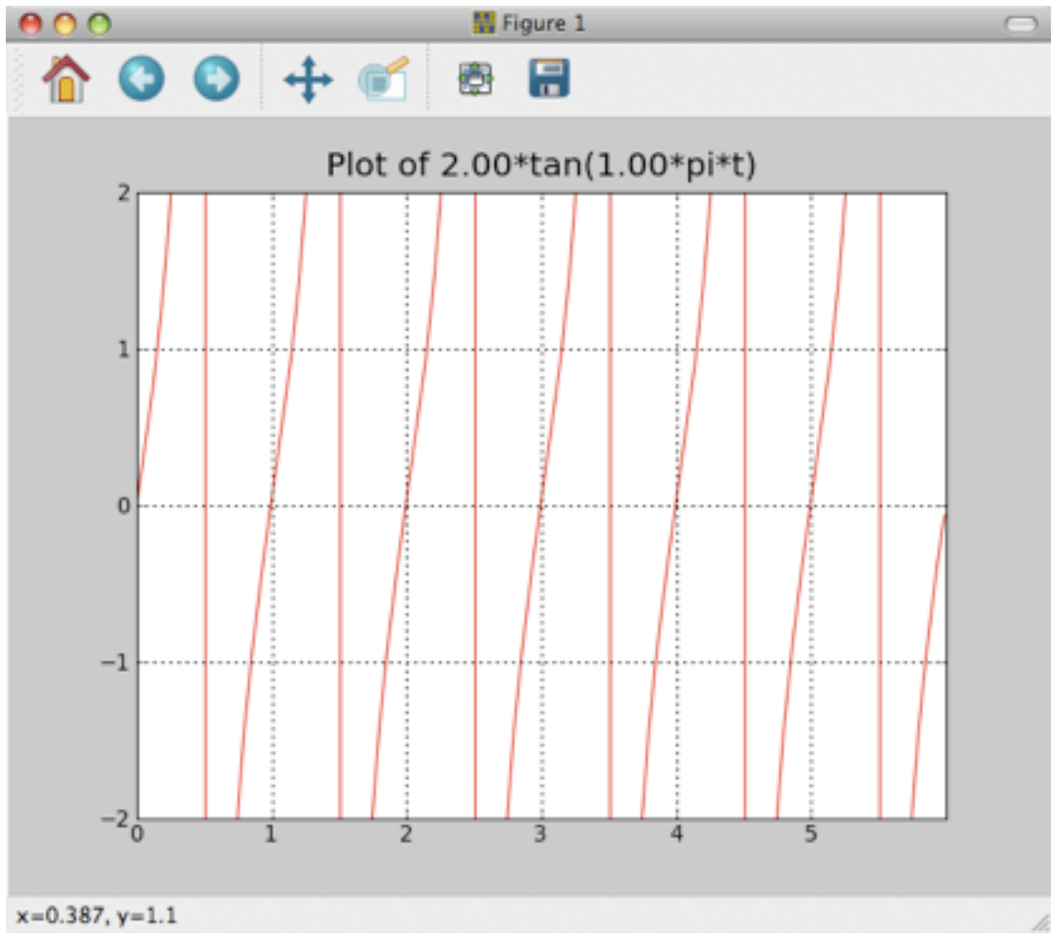
```
1  #!/usr/bin/python
2  import numpy as np
3  import matplotlib
4  from matplotlib import rc
5
6  matplotlib.use('Qt4Agg')
7
8
9  import matplotlib.pyplot as plt
10 import matplotlib.transforms as mtransforms
11 amp = float(raw_input("Enter a value for amplitude >"))
12 freq = float(raw_input("Enter a value for frequency >"))
13 t = np.arange(0.0, 6.0, 0.01)
14 plt.figure(1)
15 # set the range in the x
16 plt.xlim(0,6)
17 # set the range in the y
18 plt.ylim(-amp,amp)
19
20 plt.xticks(range(0,6))
21 plt.yticks(range(-amp,amp))
22 plt.axis([0,6,-amp,amp])
23 plt.grid(True)
24 plt.title('Plot of %0.2f*sin(%0.2f*pi*t)'%(amp,freq),
       fontsize=18)
25 plt.plot(t, amp*np.sin(freq*np.pi*t), 'r')
26
27 plt.show()
```

Cos.py



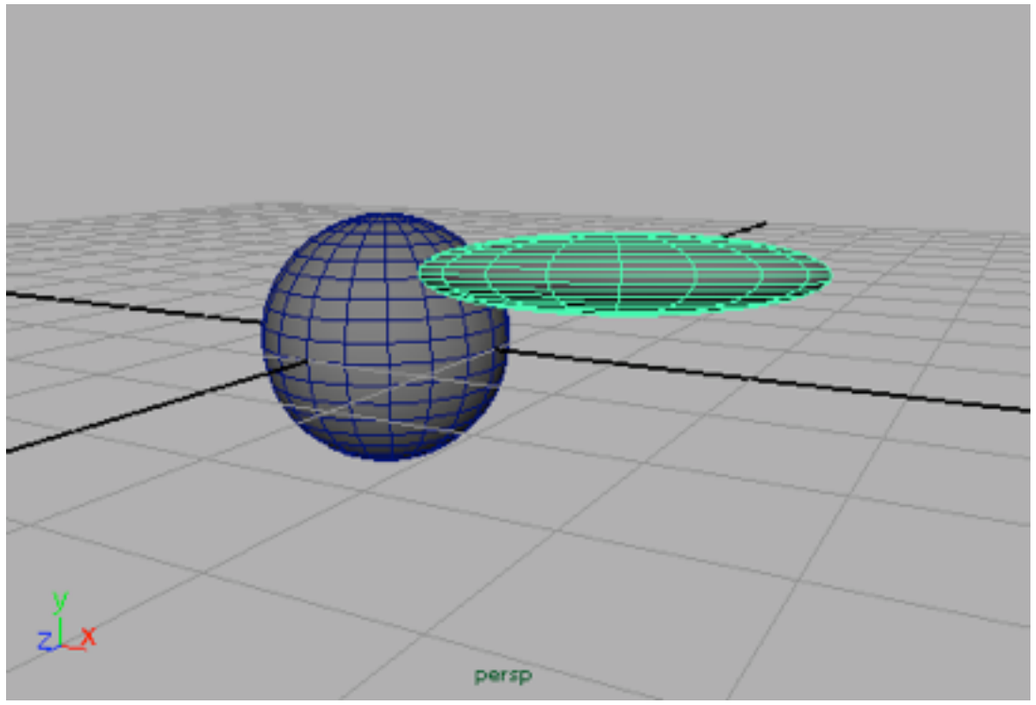

```
1  #!/usr/bin/python
2  import numpy as np
3  import matplotlib
4  from matplotlib import rc
5
6  matplotlib.use('Qt4Agg')
7
8
9  import matplotlib.pyplot as plt
10 import matplotlib.transforms as mtransforms
11 amp = float(raw_input("Enter a value for amplitude >"))
12 freq = float(raw_input("Enter a value for frequency >"))
13 t = np.arange(0.0, 6.0, 0.01)
14 plt.figure(1)
15 # set the range in the x
16 plt.xlim(0,6)
17 # set the range in the y
18 plt.ylim(-amp,amp)
19
20 plt.xticks(range(0,6))
21 plt.yticks(range(-amp,amp))
22 plt.axis([0,6,-amp,amp])
23 plt.grid(True)
24 plt.title('Plot of %0.2f*cos(%0.2f*pi*t)'%(amp,freq),
       fontsize=18)
25 plt.plot(t, amp*np.cos(freq*np.pi*t), 'r')
26
27 plt.show()
```

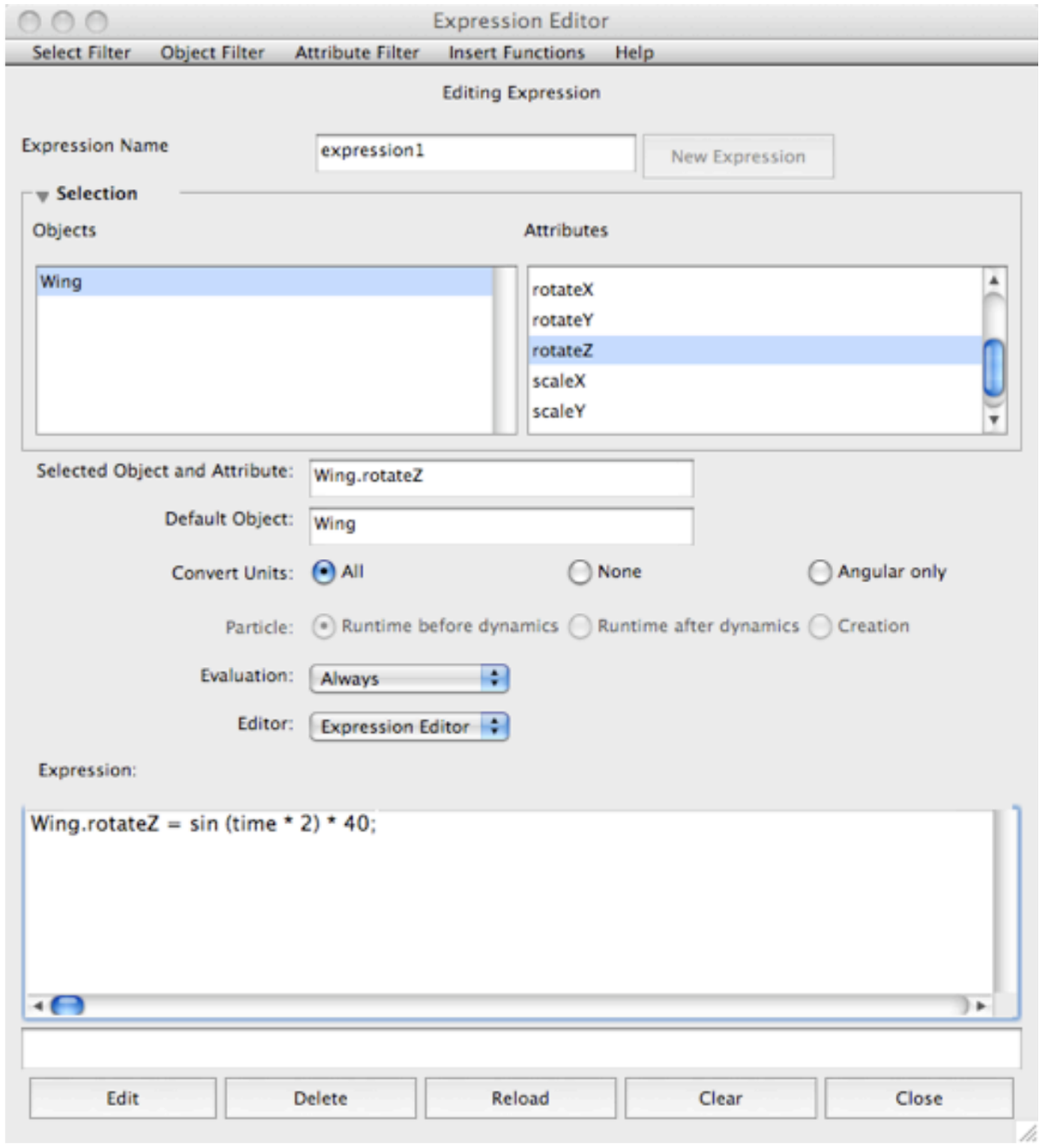
Tan.py



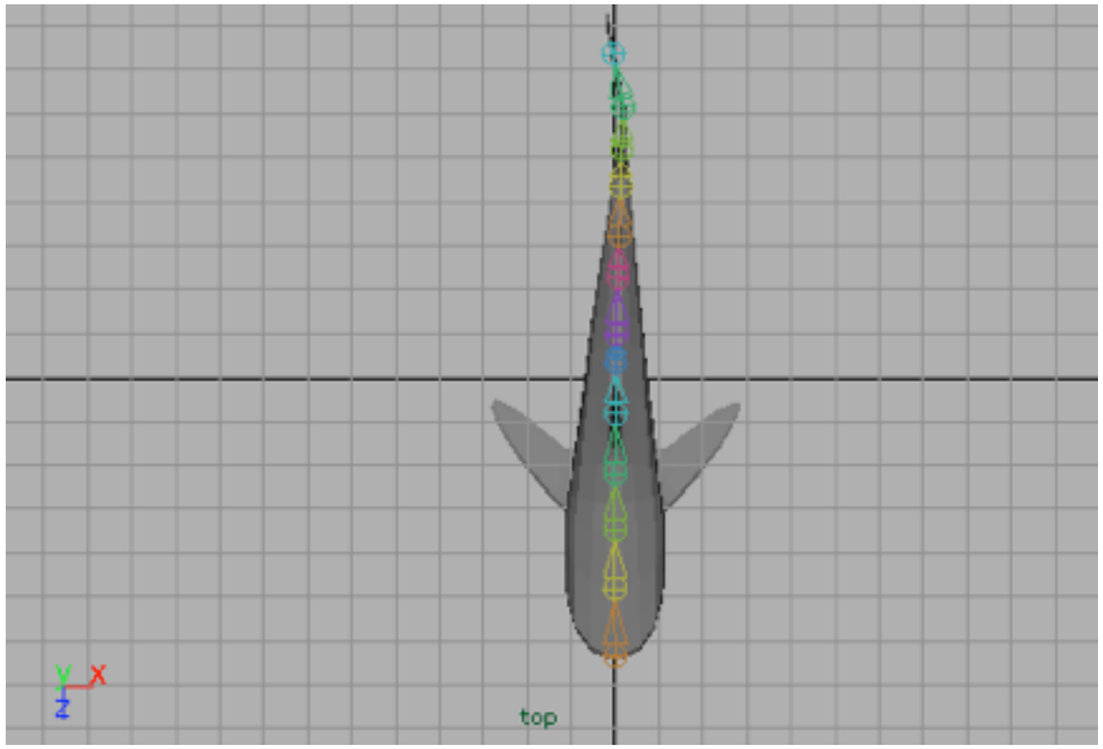

```
1  #!/usr/bin/python
2  import numpy as np
3  import matplotlib
4  from matplotlib import rc
5
6  matplotlib.use('Qt4Agg')
7
8
9  import matplotlib.pyplot as plt
10 import matplotlib.transforms as mtransforms
11 amp = float(raw_input("Enter a value for amplitude >"))
12 freq = float(raw_input("Enter a value for frequency >"))
13 t = np.arange(0.0, 6.0, 0.01)
14 plt.figure(1)
15 # set the range in the x
16 plt.xlim(0,6)
17 # set the range in the y
18 plt.ylim(-amp,amp)
19
20 plt.xticks(range(0,6))
21 plt.yticks(range(-5,5))
22 plt.axis([0,6,-amp,amp])
23 plt.grid(True)
24 plt.title('Plot of %0.2f*tan(%0.2f*pi*t)'%(amp,freq),
       fontsize=18)
25 plt.plot(t, amp*np.tan(freq*np.pi*t), 'r')
26
27 plt.show()
```

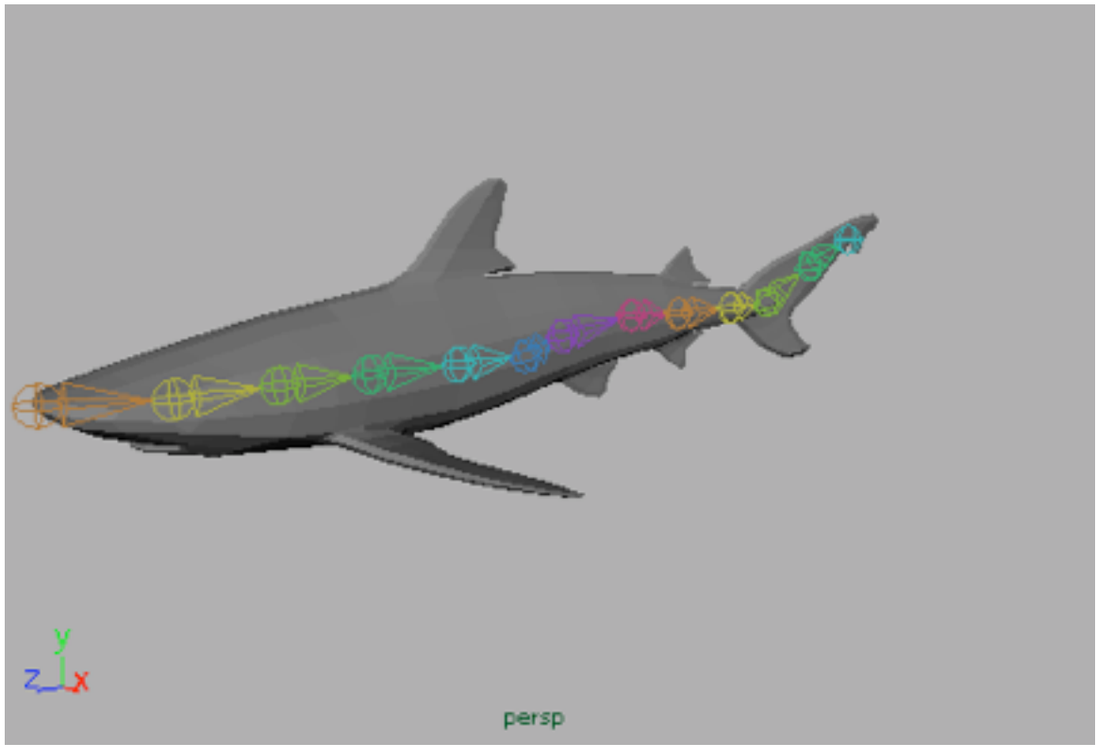
Using sine for animation





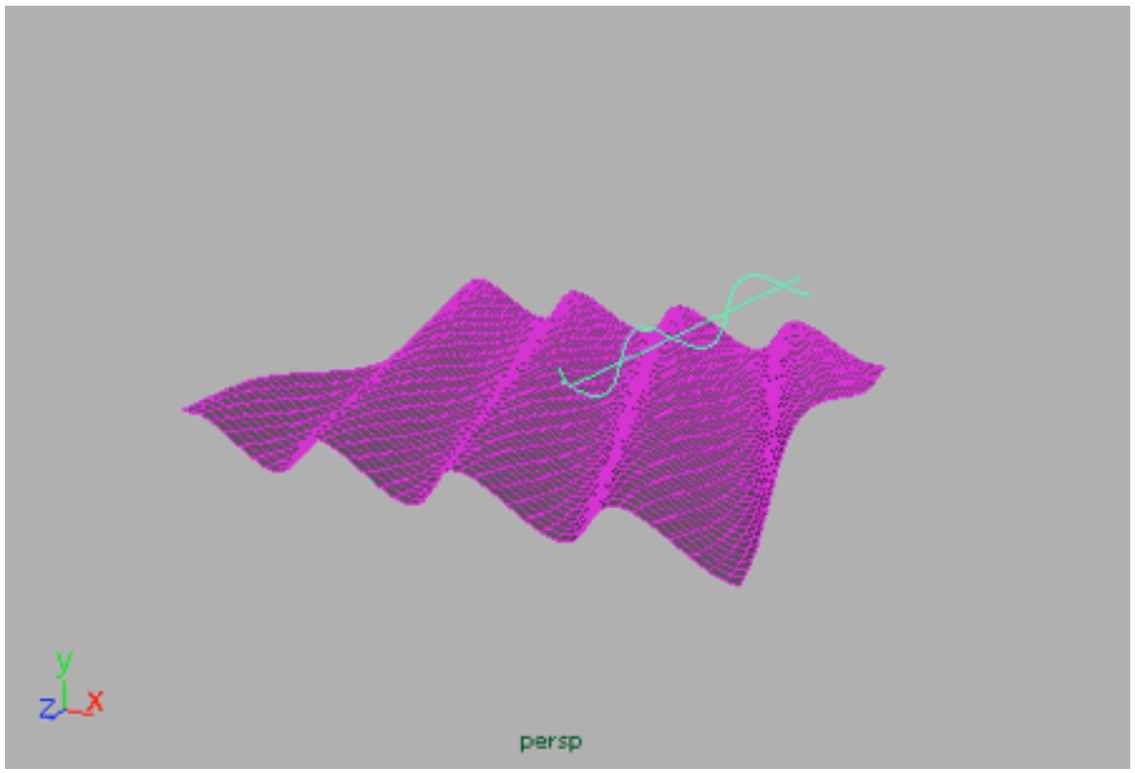
Fish

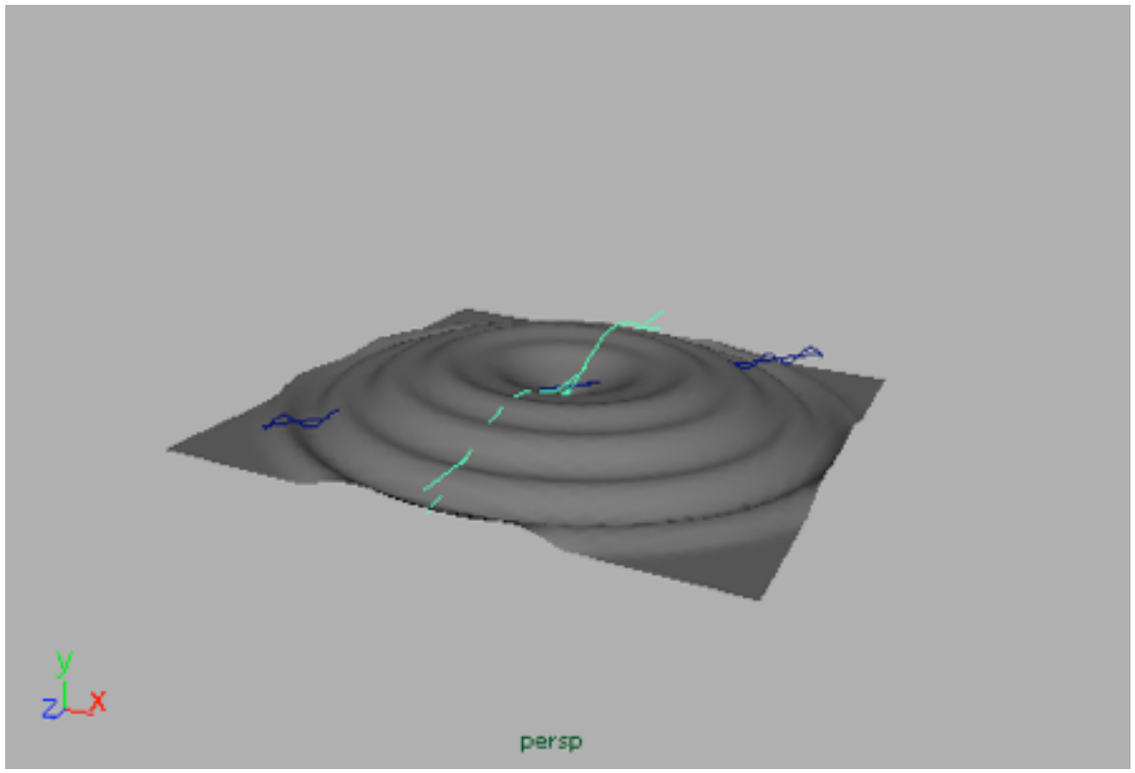




- The fish animation has a number of sine / cosine expressions for the rotation of the ik joints
- By using a combination of -sine(time) and sine(time) we get the swimming motion

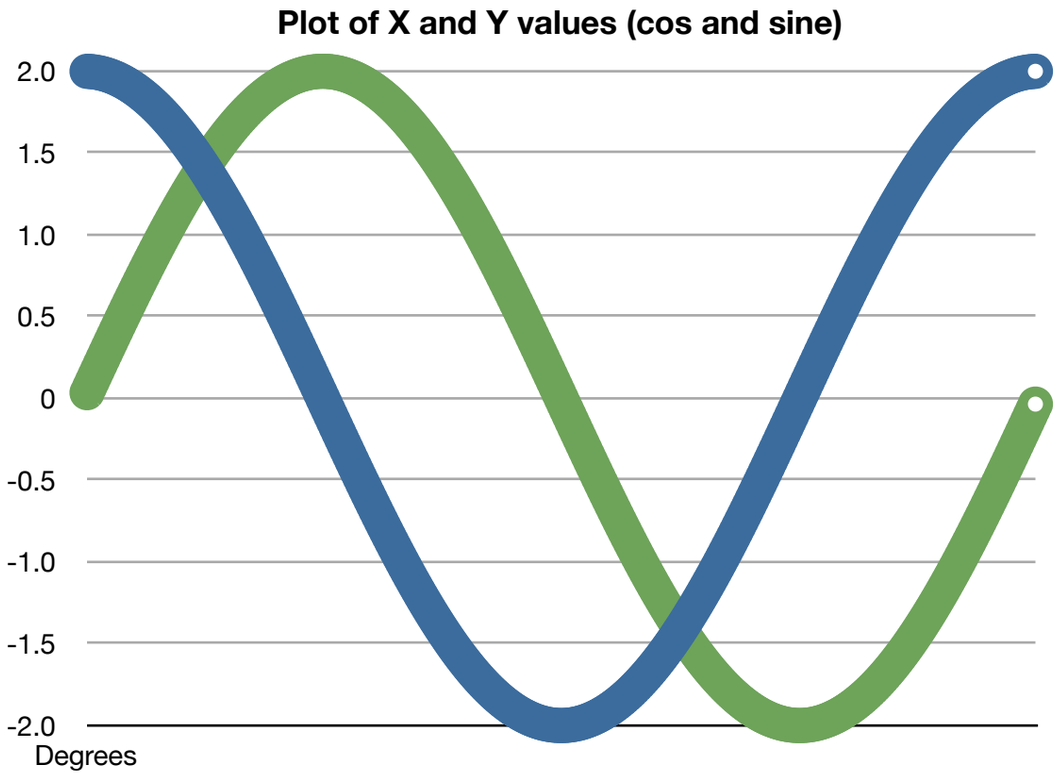
Sine Deformation of Grid





Generating Circles

- To generate the points on a circle we can use cos and sine in the following way




```
#!/usr/bin/python
from math import sin,cos,radians,degrees

radius=float(raw_input("the␣radius␣>"))

for angle in range(0,360) :
  x = radius*cos(radians(angle))
  y = radius*sin(radians(angle))
  print "%f\t%f" %(x,y)
```

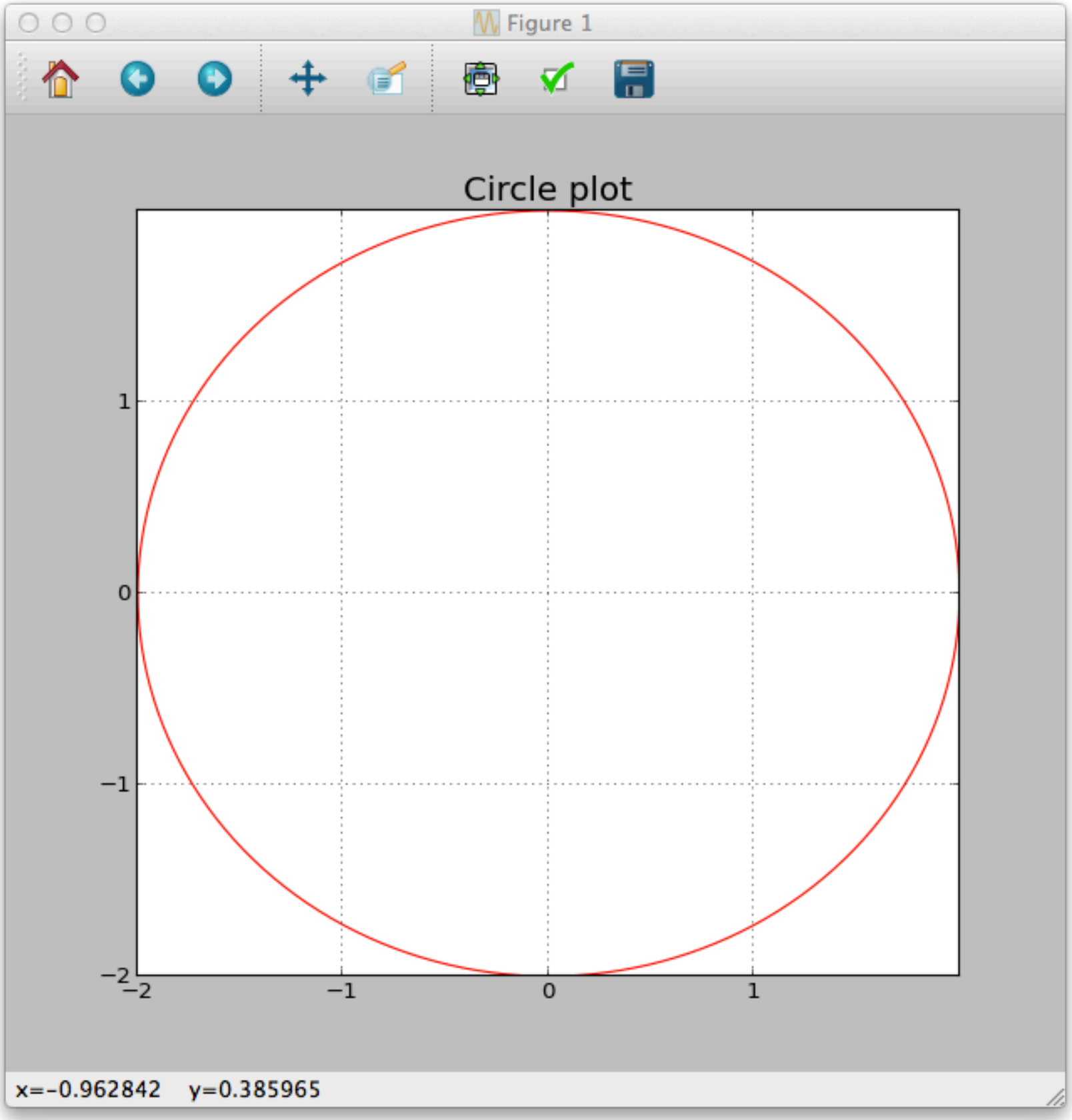
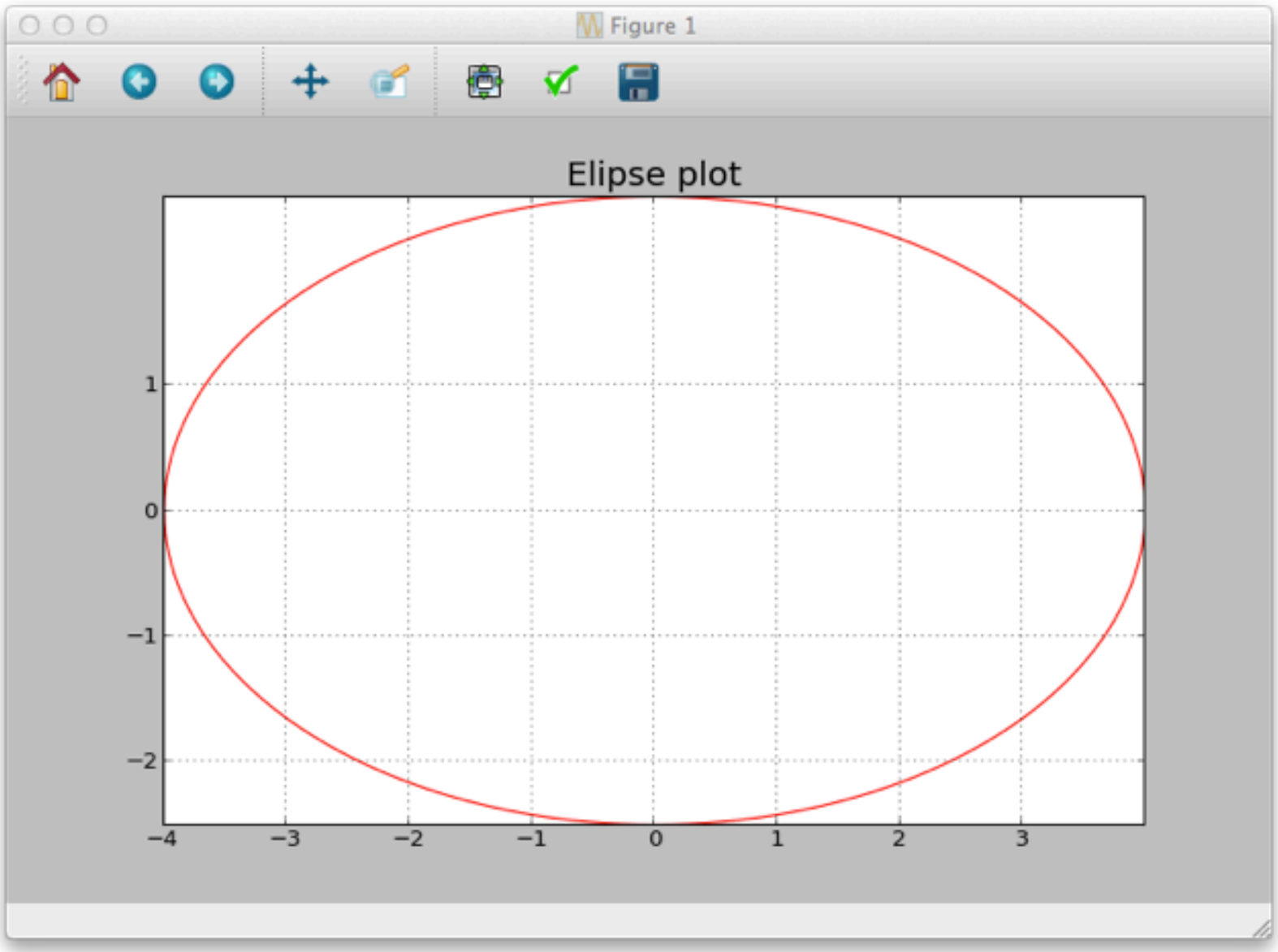


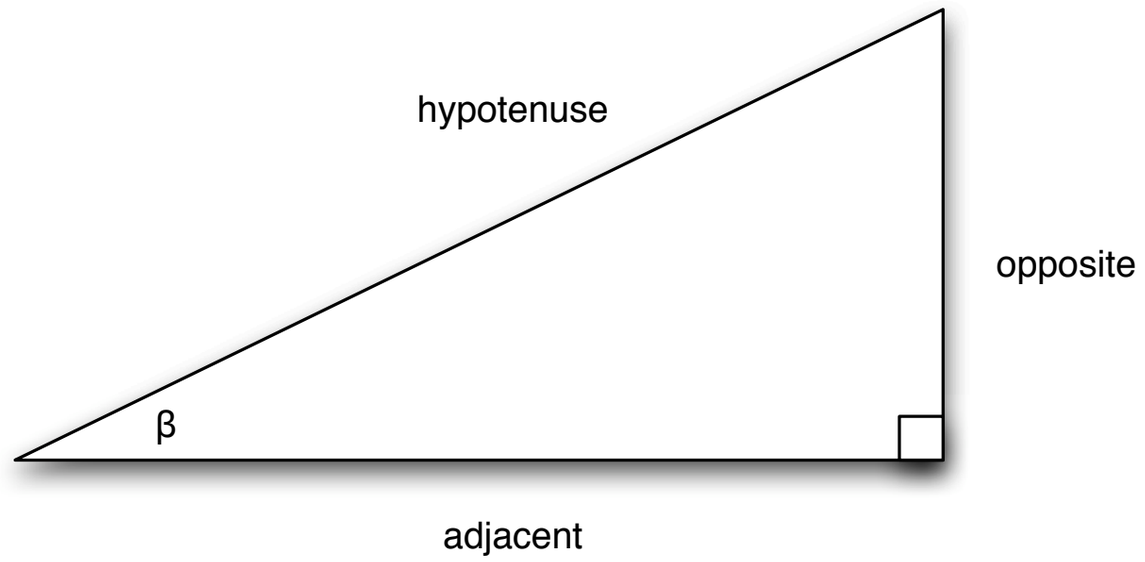
Figure 1
Circle plot
1
0
−1
−2
−2
−1
0
1
x=-0.962842 y=0.385965

An Ellipse

- Is just a circle where we have a different value for the radius in x and y



The Trigonometric ratios



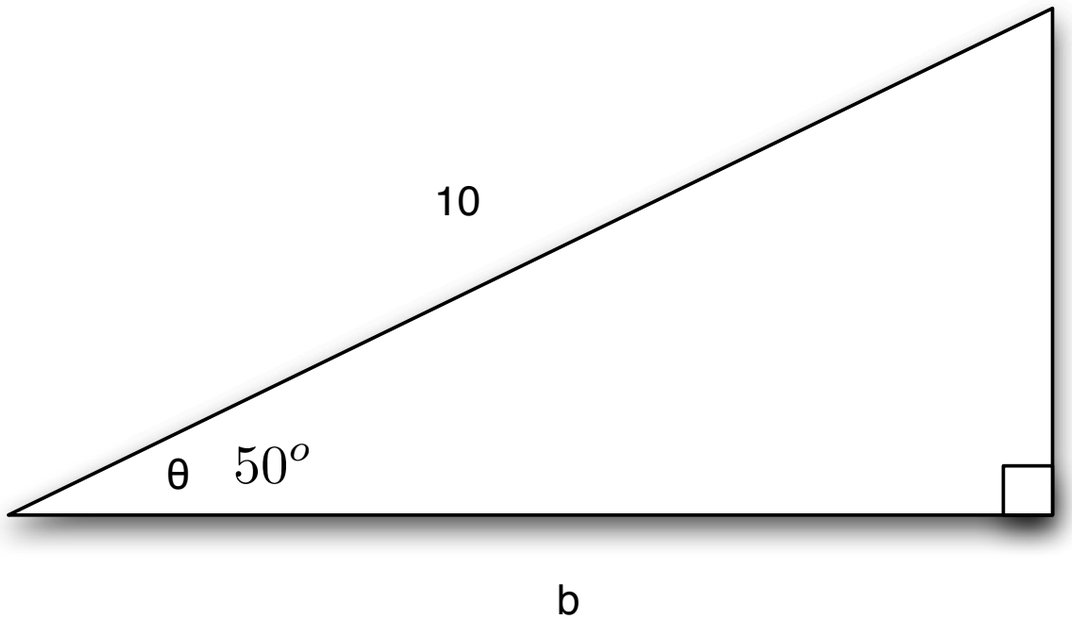
$$\sin(\beta) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos(\beta) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan(\beta) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\csc(\beta) = \frac{1}{\sin(\beta)} \qquad \sec(\beta) = \frac{1}{\cos(\beta)} \qquad \cot(\beta) = \frac{1}{\tan(\beta)}$$

Mnemonics to help remember Trig Relationships

- There are a number of ways to remember these relationships and many can be found at http://en.wikipedia.org/wiki/Trigonometry_mnemonics
- Some Old Horses Chase And Hunt Till Old Age
- Some Old Hippie Caught A High Tripping On Acid
- Signs Of Happiness Come After Having Tubs Of Acid
- Some Old Hags Can't Always Hide Their Old Age
- Some Old Horses Can Always Hear Their Owner's Approach
- Silly Old Harry Caught A Herring Trawling Off America
- SOHCAHTOA (sounds like "soak a toe-a", can be read as "soccer tour")

Example



$$\frac{h}{10} = \sin(50^o)$$
$$h = 10\sin(50^o) = 10 \times 0.76604$$
$$h = 7.66$$

$$\frac{b}{10} = \cos(50^o)$$
$$b = 10\cos(50^o) = 10 \times 0.64279$$
$$b = 6.4279$$

Trig in Scripts

- Virtually all programming languages treat the cos, sine and tan function using radians for the angle measurement.
- This means we need to convert to radians when using them

```
1  #!/usr/bin/python
2  from math import sin,asin,radians,degrees
3
4  beta=float(raw_input("Enter␣an␣Angle␣in␣degrees␣>"))
5
6  print beta, "␣in␣radians␣=␣",radians(beta)
7  rad=radians(beta)
8  sbeta=sin(rad)
9  print "sin(beta)␣=␣",sbeta
10 print "asin(beta)␣=␣",degrees(asin(sbeta))
```

Inverse trigonometric ratios

- The sin, cos and tan functions convert angles into ratios, and the inverse functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ convert ratios into angles

- For example,

$$\sin(45^o) = 0.707, \text{ therefore } \sin^{-1}(0.707) = 45^o$$

- Although sine and cosine functions are cyclic functions (i.e. they repeat indefinitely) the inverse functions return angles over a specific period

$$\sin^{-1}(x) = \theta \text{ where } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \text{ and } \sin(\theta) = x$$
$$\cos^{-1}(x) = \theta \text{ where } 0 \leq \theta \leq \pi \text{ and } \cos(\theta) = x$$
$$\tan^{-1}(x) = \theta \text{ where } -\pi \leq \theta \leq \frac{\pi}{2} \text{ and } \tan(\theta) = x$$

Trigonometric Relationships

- There is an intimate relationship between the sin and cos definitions and are formally related by

$$\cos(\beta) = \sin(\beta + 90^o)$$

- The Theorem of Pythagoras can be used to derive other formulae such as

$$\frac{\sin(\beta)}{\cos(\beta)} = \tan(\beta)$$

$$\sin^2(\beta) + \cos^2(\beta) = 1$$

$$1 + \tan^2(\beta) = \sec^2(\beta)$$

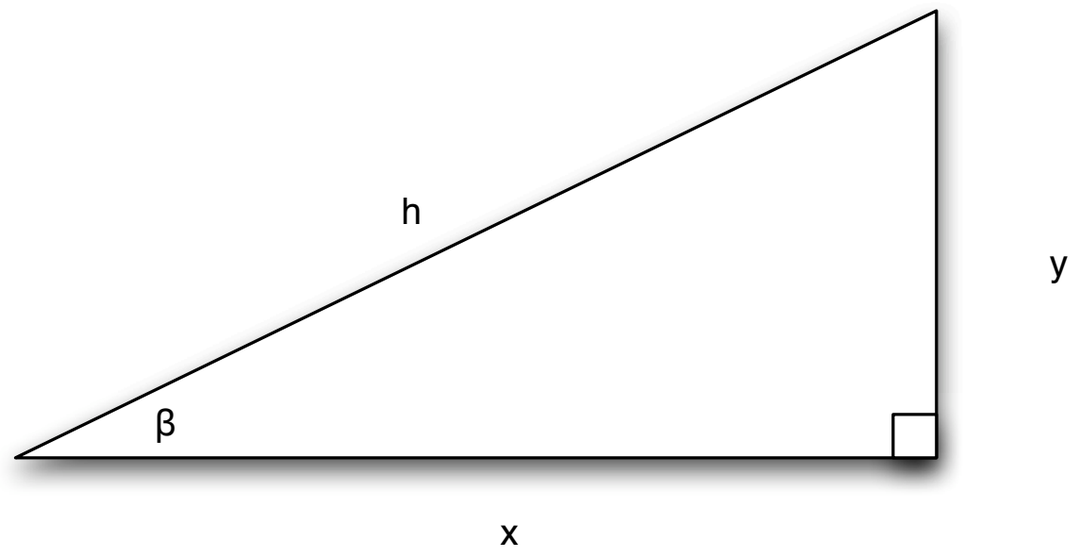
$$1 + \cot^2(\beta) = cosec^2(\beta)$$

TrigRatios.py

```
[jmacey@neuromancer:Lecture3]$./TrigRatio.py
Enter an Angle in degrees >20
The Trigonometric Ratios

 sin(20.00) = 0.342020 cos(20.00) = 0.939693 tan(20.00) = 0.363970
sin(beta)/cos(bets) = 0.363970 = tan(beta) = 0.363970
sin^2(beta) + cos^2(beta) = 1.000000
1+tan^2(beta) = 1.132474 = sec^2(beta) = 1/cos(beta) ^2 = 1.132474
1+cot^2(beta) = 8.548632 = cosec^2(beta) = 1/tan(beta) ^2 = 8.548632
```

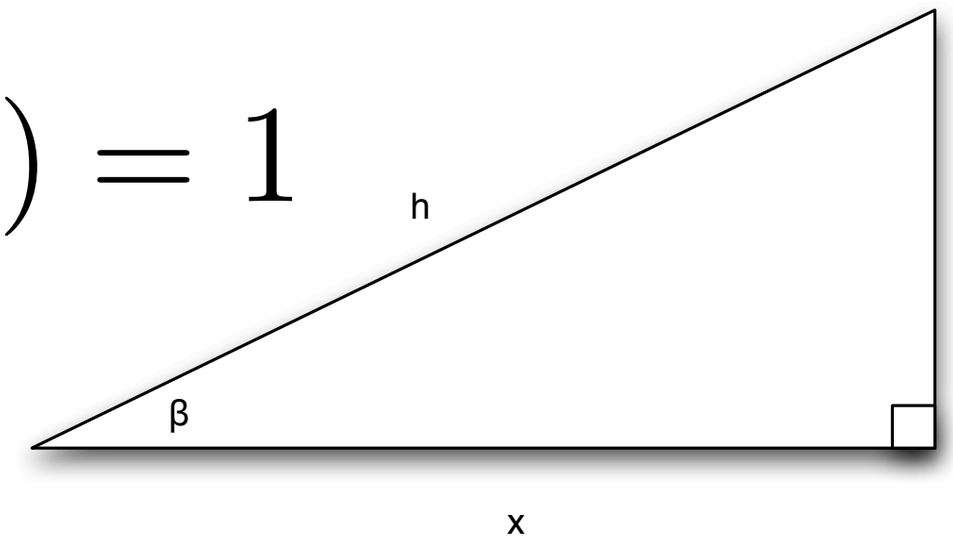
```
1  #!/usr/bin/python
2  from math import cos,acos,tan,atan,sin,asin,radians,degrees
3
4  beta=float(raw_input("Enter an Angle in degrees >"))
5
6  rad=radians(beta)
7
8  sbeta=sin(rad)
9  cbeta=cos(rad)
10 tbeta=tan(rad)
11 secbeta=1/cbeta
12 cotbeta=1/tbeta
13 cosecbeta=1/sbeta
14 print "The Trigonometric Ratios"
15 print "\n sin(%.2f) = %f cos(%.2f) = %f tan(%.2f) = %f"%(beta,sbeta,beta,cbeta,beta,tbeta)
16 print "sin(beta)/cos(bets) = %f = tan(beta) = %f" %(sbeta/cbeta,tbeta)
17 print "sin^2(beta) + cos^2(beta) = %f" %(sbeta*sbeta+cbeta*cbeta)
18 print "1+tan^2(beta) = %f = sec^2(beta) = 1/cos(beta) ^2 = %f" %(1+(tbeta*tbeta),secbeta*
       secbeta)
19 print "1+cot^2(beta) = %f = cosec^2(beta) = 1/tan(beta) ^2 = %f" %(1+(cotbeta*cotbeta),
       cosecbeta*cosecbeta)
```

Proof $\dfrac{\sin(\beta)}{\cos(\beta)} = \tan$



$$\sin(\beta) = \frac{y}{h}$$
$$\cos(\beta) = \frac{x}{h}$$
$$\frac{\sin(\beta)}{\cos(\beta)} = \frac{yh}{hx} = \frac{y}{x} = \tan(\beta)$$
$$\frac{\sin(\beta)}{\cos(\beta)} = \tan(\beta)$$

Proof $\sin^2(\beta) + \cos^2(\beta) = 1$



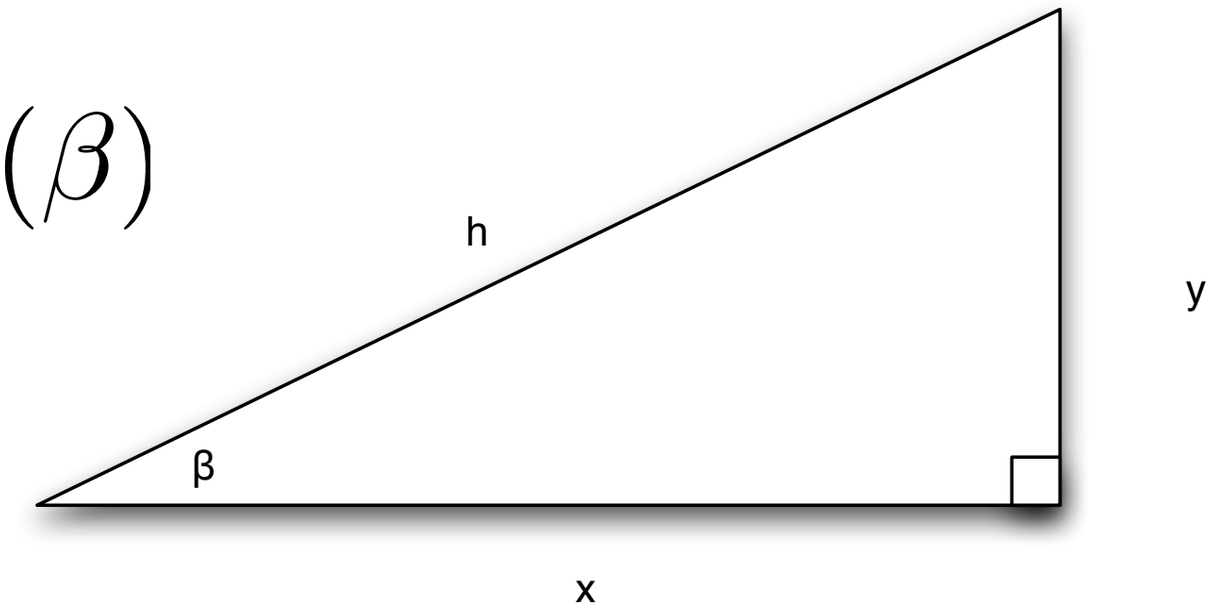
$$y^2 + x^2 = h^2$$

$$\frac{y^2}{h^2} + \frac{x^2}{h^2} = \frac{h^2}{h^2}$$

$$(\frac{y}{h})^2 + (\frac{x}{h})^2 = 1$$

$$\sin^2(\beta) + cos^2(\beta) = 1$$

Proof $1+\tan^2(\beta) = \sec^2(\beta)$

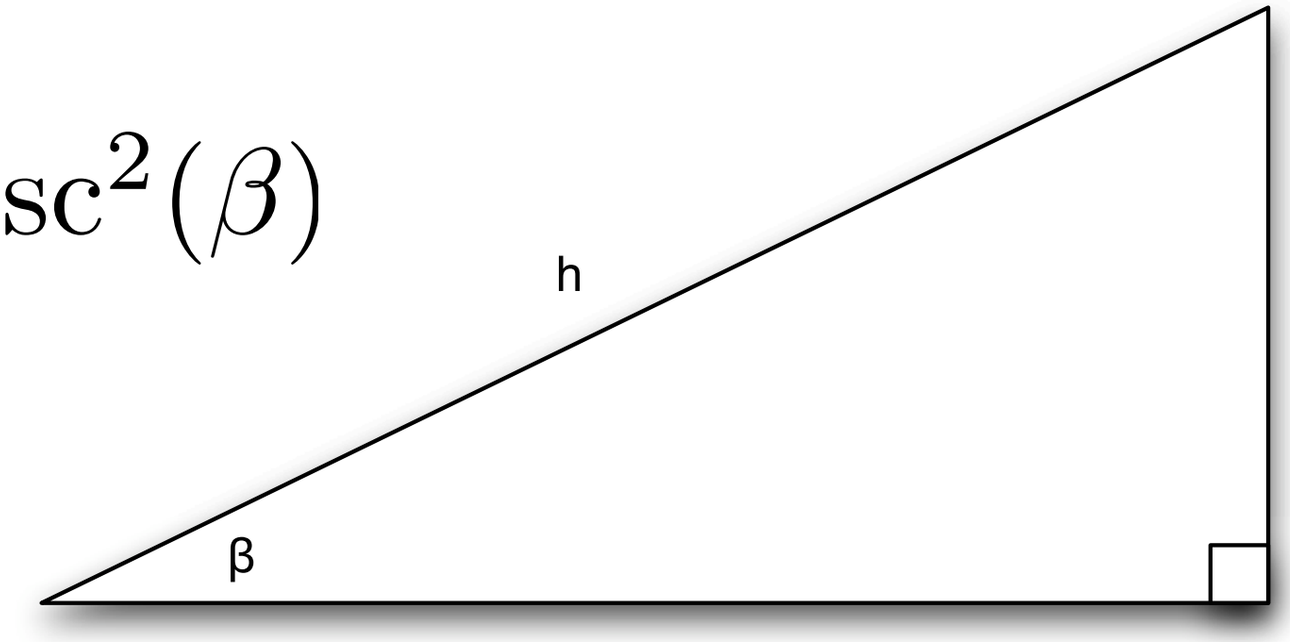


$$\cos^2(\beta) + \sin^2(\beta) = 1$$

$$\frac{\cos^2(\beta)}{\cos^2(\beta)} + \frac{\sin^2(\beta)}{\cos^2(\beta)} = \frac{1}{\cos^2(\beta)}$$

$$1+\tan^2(\beta) = \sec^2(\beta)$$

Proof $1+\cot^2(\beta) = \csc^2(\beta)$



$$\sin^2(\beta) + \cos^2(\beta) = 1$$

$$\frac{\sin^2(\beta)}{\sin^2(\beta)} + \frac{\cos^2(\beta)}{\sin^2(\beta)} = \frac{1}{\sin(\beta)}$$

$$1+\cot^2(\beta) = \csc^2(\beta)$$

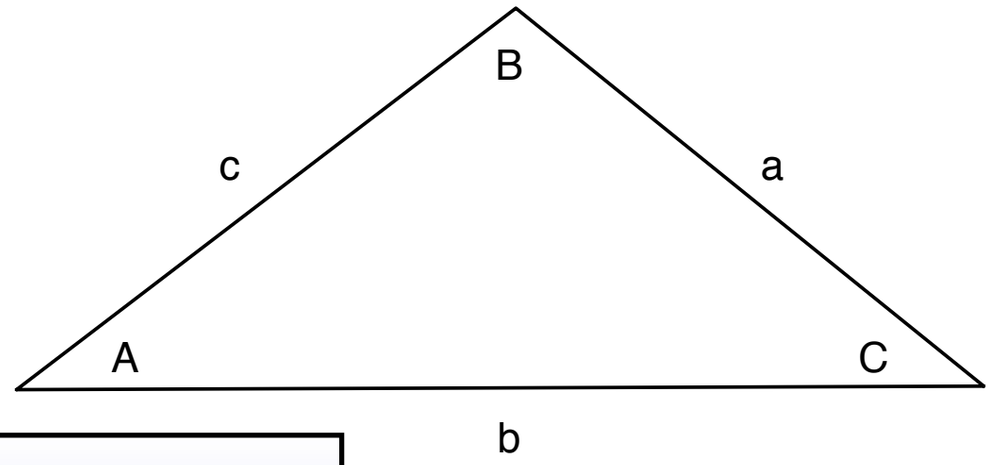
The Law of Sines

- The law of sines for an arbitrary triangle states:

$$\frac{a}{\sin(a)} = \frac{b}{\sin(b)} = \frac{c}{\sin(c)} = 2r$$

- It can be proven by dividing the triangle into two right ones and using the above definition of sine.
- The common number $\frac{a}{\sin(a)}$ occurring in the theorem is the reciprocal of the diameter of the circle through the three points A, B and C.
- The law of sines is useful for computing the lengths of the unknown sides in a triangle if two angles and one side are known.
- This is a common situation occurring in triangulation, a technique to determine unknown distances by measuring two angles and an accessible enclosed distance.

The Sine Rule



$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

Example

$$\text{A}{=}50^o, B = 30^o, a = 10, \text{ find b}$$

$$\frac{b}{\sin(30)} = \frac{10}{\sin(50)}$$

$$\therefore b = \frac{10\sin(30)}{\sin(50)} = \frac{5}{0.766} = 6.5274$$

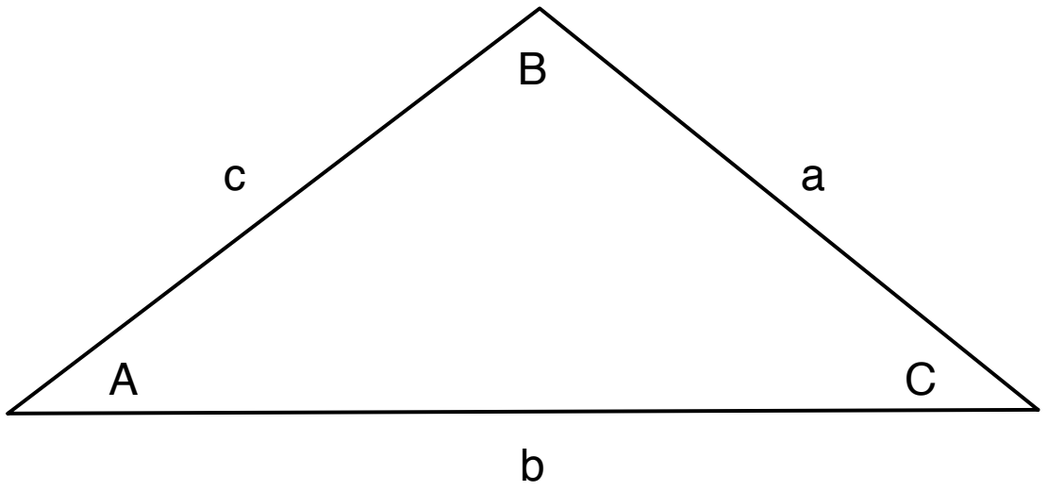
The Law of Cosines

- The law of cosines is an extension of the Pythagorean theorem:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

- Again, this theorem can be proven by dividing the triangle into two right ones.
- The law of cosines is useful to determine the unknown data of a triangle if two sides and an angle are known.
- If the angle is not contained between the two sides, the triangle may not be unique.
- Be aware of this ambiguous case of the Cosine law.

The Cosine Rule



$$\mathrm{a}^2 = b^2 + c^2 - 2bc\cos(A)$$

$$\mathrm{b}^2 = c^2 + a^2 - 2ca\cos(B)$$

$$\mathrm{c}^2 = a^2 + b^2 - 2ab\cos(C)$$

$$\mathrm{a}{=}\mathrm{b}\cos(C) + c\cos(B)$$

$$\mathrm{b}{=}\mathrm{c}\cos(A) + a\cos(C)$$

$$\mathrm{c}{=}\mathrm{a}\cos(B) + b\cos(A)$$

The Law of Tangents

- There is also a law of Tangents but is less commonly used

$$\frac{a+b}{a-b} \equiv \frac{\tan\left[\frac{1}{2}(A+B)\right]}{\tan\left[\frac{1}{2}(A-B)\right]}$$

Compound Angles

- Two sets of compound trigonometric relationships show how to add and subtract two different angles and multiples of the same angle.
- The following are some of the most common relationships

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)sin(B)$$

$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

$$\sin(2\beta) = 2\sin(\beta)\cos(\beta)$$

$$\cos(2\beta) = \cos^2(\beta) - \sin^2(\beta)$$

$$\cos(2\beta) = 2\cos^2(\beta) - 1$$

$$\cos(2\beta) = 1 - 2\sin^2(\beta)$$

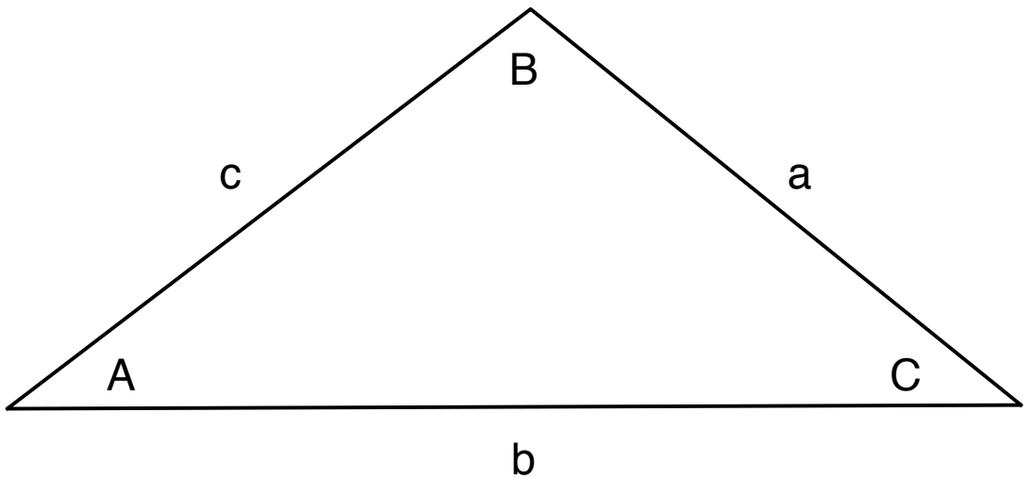
$$\sin(3\beta) = 3\sin(\beta) - 4\sin^3(\beta)$$

$$\cos(3\beta) = 4\cos^3(\beta) - 3\cos(\beta)$$

$$\cos^2(\beta) = \frac{1}{2}(1 + \cos(2\beta))$$

$$\sin^2(\beta) = \frac{1}{2}(1 - \cos(2\beta))$$

Perimeter Relationships



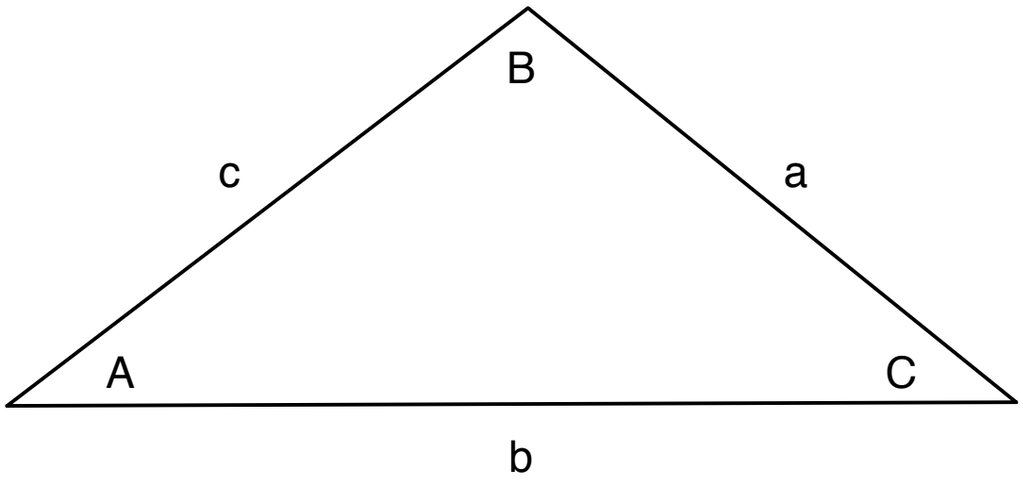
$$s = \frac{1}{2}(a+b+c)$$

$$\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\sin\left(\frac{B}{2}\right) = \sqrt{\frac{(s-c)(s-a)}{ca}}$$

$$\sin\left(\frac{C}{2}\right) = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

Perimeter Relationships



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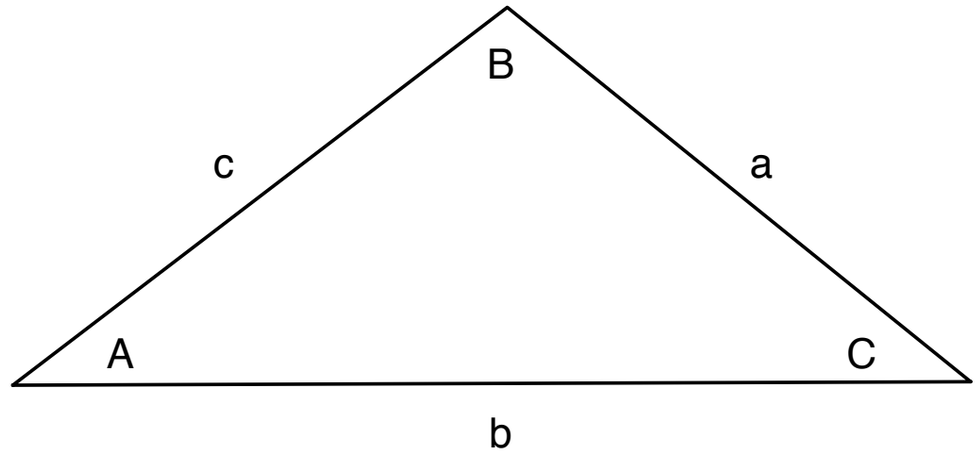
Perimeter Relationships



$$s = \frac{1}{2}(a+b+c)$$

$$\sin(A) = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

$$\sin(B) = \frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)}$$

$$\sin(C) = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$

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